Assignment 23 solutions

1.

$$\text{Area} = \int_0^8 y^{\frac{1}{3}}\,\mathrm{d}y = \left.\frac{3y^{\frac{4}{3}}}{4}\right|_0^8 = 12 \text{ units}^2$$

$$\text{Volume} = \int_0^8 \pi y^{\frac{2}{3}}\,\mathrm{d}y = \left.\frac{3y^{\frac{5}{3}}}{5}\pi\right|_0^8 = \frac{96\pi}{5} \text{ units}^3$$

2.

$x = y^2 + 1 \rightarrow (1)$

$x = 4 - 2y \rightarrow (2)$

The points of intersection are $(2,1)$ and $(10,-3)$

$$\text{Area} = \frac{1}{2}\times 2\times 1 + \int_0^1 y^2 + 1\,\mathrm{d}y = 1 + \left[\frac{y^3}{3} + y\right]_0^1 = \frac{7}{3} \text{ units}^2$$

$$\text{Volume} = \int_0^2 \pi\left(\frac{4-x}{2}\right)^2 \mathrm{d}x - \int_1^2 \pi(x-1)\,\mathrm{d}x$$

$$= \frac{1}{4}\pi\left[\frac{(4-x)^3}{-3}\right]_0^2 - \pi\left.\left(\frac{x^2}{2} - x\right)\right|_1^2$$

$$= \frac{1}{4}\pi\left(-\frac{1}{3}\right)(8-64) - \frac{1}{2}\pi$$

$$= \frac{25}{6}\pi \text{ units}^3$$

3.

$$\text{Area} = \left|\int_1^3 (y-2)^2 - 1\,\mathrm{d}y\right| + \int_0^1 (y-2)^2 - 1\,\mathrm{d}y$$

$$= \left|\int_1^3 y^2 - 4y + 3\,\mathrm{d}y\right| + \int_0^1 y^2 - 4y + 3\,\mathrm{d}y$$

$$= \left|\left[\frac{y^3}{3} - 2y^2 + 3y\right]_1^3\right| + \left[\frac{y^3}{3} - 2y^2 + 3y\right]_0^1$$

$$= \left|0 - \frac{4}{3}\right| + \frac{4}{3}$$

$$= \frac{8}{3} \text{ units}^2$$

$$\text{Volume} = \int_0^3 \pi\left((y-2)^2 - 1\right)^2 \mathrm{d}y$$

$$= \pi\int_0^3 (y-2)^4 - 2(y-2)^2 + 1\,\mathrm{d}y$$

$$= \pi\left[\frac{(y-2)^5}{5} - \frac{2(y-2)^3}{3} + y\right]_0^3$$

$$= \pi\left(\frac{38}{15} - \left(-\frac{16}{15}\right)\right)$$

$$= \frac{18}{5}\pi \text{ units}^3$$

4(i).

$$\text{Area} = \int_0^{1.5}\left(e^x - 1\right)\mathrm{d}x - \int_0^{\ln 2.5}(1.5) - \left(e^y - 1\right)\mathrm{d}y$$

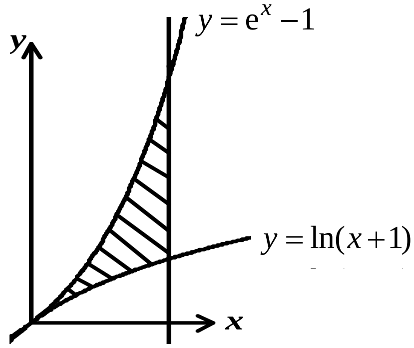
$$= \left[e^x - x\right]_0^{1.5} - \int_0^{\ln 2.5} 2.5 - e^y\,\mathrm{d}y$$

$$= \left[e^{1.5} - 1.5 - 1\right] - \left[2.5y - e^y\right]_0^{\ln 2.5}$$

$$= \left[e^{1.5} - 2.5\right] - \left[2.5(\ln 2.5) - e^{\ln 2.5} + e^0\right]$$

$$= e^{1.5} - 2.5\ln 2.5 - 1$$

$$= 1.19$$



4(ii). [x-axis]

$$\text{Volume} = \int_0^{1.5} \pi\left(e^x-1\right)^2 - \pi\left(\ln\left(x+1\right)\right)^2 \ dx$$
$$= \pi\left[\int_0^{1.5} e^{2x} - 2e^x + 1 \, dx - \int_0^{1.5}\left(\ln\left(x+1\right)\right)^2 \ dx\right]$$

$$\int_0^{1.5}\left(\ln\left(x+1\right)\right)^2 \ dx = \pi\int\left(\ln u\right)^2 \ du$$ use substitution, where $u = x+1 \to \frac{du}{dx} = 1$

$$= \pi\left[u\left(\ln u\right)^2 - \int u \cdot 2\ln u \cdot \frac{1}{u}\, du\right]$$ by parts where $'u' = \left(\ln u\right)^2, 'dv' = 1$

$$= \pi\left[u\left(\ln u\right)^2 - \int 2\ln u \ du\right]$$
$$= \pi\left[u\left(\ln u\right)^2 - 2\left(u\ln u - u\right)\right]$$
$$= \pi\left[u\left(\ln u\right)^2 - 2u\ln u + 2u\right]$$
$$= \pi\left[\left(x+1\right)\left(\ln\left(x+1\right)\right)^2 - 2\left(x+1\right)\ln\left(x+1\right) + 2\left(x+1\right)\right]$$

$$\pi\int_0^{1.5}\left(\ln\left(x+1\right)\right)^2 \ dx = \pi\left[\left(x+1\right)\left(\ln\left(x+1\right)\right)^2 - 2\left(x+1\right)\ln\left(x+1\right) + 2\left(x+1\right)\right]_0^{1.5}$$
$$= \pi\left[\left(2.5\left(\ln 2.5\right)^2 - 2\left(2.5\right)\ln 2.5 + 2\left(2.5\right)\right) - \left(0 - 0 + 2\right)\right]$$
$$\approx 0.51752\pi$$

s

$$\pi\left[\int_0^{1.5} e^{2x} - 2e^x + 1\, dx - \int_0^{1.5}\left(\ln\left(x+1\right)\right)^2 \ dx\right] = \pi\left[\frac{e^{2x}}{2} - 2e^x + x\right]_0^{1.5} - 0.51752\pi$$
$$= 4.0794\pi - 0.51752\pi$$
$$\approx 3.56\pi \text{ units}^3$$

4(ii). [y-axis]

Volume = cylinder $- A - B$

$$= \pi(1.5)^2\left(e^{1.5}-1\right) - \int_0^{e^{1.5}-1} \pi\left(\ln(y+1)\right)^2\, dy - \int_0^{\ln 2.5} \pi\left[(1.5)^2 - \left(e^y - 1\right)^2\right] dy$$

$$\pi\int\left(\ln(y+1)\right)^2\, dy = \pi\int(\ln u)^2\, du \qquad \text{use substitution, where } u = y+1 \to \frac{du}{dy} = 1$$

$$= \pi\left[u(\ln u)^2 - \int u\cdot 2\ln u\cdot\frac{1}{u}\, du\right] \quad \text{by parts where } 'u' = (\ln u)^2, 'dv' = 1$$

$$= \pi\left[u(\ln u)^2 - \int 2\ln u\, du\right]$$

$$= \pi\left[u(\ln u)^2 - 2(u\ln u - u)\right]$$

$$= \pi\left[u(\ln u)^2 - 2u\ln u + 2u\right]$$

$$= \pi\left[(y+1)\left(\ln(y+1)\right)^2 - 2(y+1)\ln(y+1) + 2(y+1)\right]$$

$$\pi\int_0^{e^{1.5}-1}\left(\ln(y+1)\right)^2\, dy = \pi\left[(y+1)\left(\ln(y+1)\right)^2 - 2(y+1)\ln(y+1) + 2(y+1)\right]_0^{e^{1.5}-1}$$

$$= \pi\left[\left(e^{1.5}(1.5)^2 - 2e^{1.5}(1.5) + 2e^{1.5}\right) - (0 - 0 + 2)\right]$$

$$\approx 3.60211\pi$$

$$\pi\int_0^{\ln 2.5}(1.5)^2 - \left(e^y - 1\right)^2\, dy = \pi\int_0^{\ln 2.5} 2.25 - \left(e^{2y} - 2e^y + 1\right) dy$$

$$= \pi\int_0^{\ln 2.5} 1.25 - e^{2y} + 2e^y\, dy$$

$$= \pi\left[1.25y - \frac{1}{2}e^{2y} + 2e^y\right]_0^{\ln 2.5}$$

$$= \pi\left[\left(1.25\ln 2.5 - \frac{1}{2}e^{2\ln 2.5} + 2e^{\ln 2.5}\right) - \left(0 - \frac{1}{2} + 2\right)\right]$$

$$= \pi\left[(1.25\ln 2.5 - 3.125 + 5) - 1.5\right]$$

$$\approx 1.52036\pi$$

Hence, Volume $= \pi(1.5)^2\left(e^{1.5}-1\right) - 5.50319\pi - 3.02036\pi$

$$= 7.83380\pi - 3.60211\pi - 1.52036\pi$$

$$\approx 2.71\pi \text{ units}^3$$

5(i).

$$1 = k + 6(-1) - 2(-1)^2$$
$$1 = k - 6 - 2$$
$$k = 9$$

$$x^2 = k + 6x - 2x^2$$
$$3x^2 - 6x - 9 = 0$$
$$x^2 - 2x - 3 = 0$$
$$(x-3)(x+1) = 0$$
$$x = 3 \text{ or } x = -1$$
$$y = 9 \text{ or } y = 1$$
$$P(3, 9)$$

5(ii).

$$\begin{aligned}\text{Area} &= \int_{-1}^{3} 9 + 6x - 2x^2 - x^2 \ \mathrm{d}x \\ &= \int_{-1}^{3} 9 + 6x - 3x^2 \ \mathrm{d}x \\ &= \left[9x + 3x^2 - x^3\right]_{-1}^{3} \\ &= (27 + 27 - 27) - (-9 + 3 - (-1)) \\ &= 27 - (-5) \\ &= 32\end{aligned}$$

$$\begin{aligned}\text{Volume} &= \pi\int_{-1}^{3} \left(9 + 6x - 2x^2\right)^2 - x^4 \ \mathrm{d}x \\ &= \pi\int_{-1}^{3} 3x^4 - 24x^3 + 108x + 81 \ \mathrm{d}x \\ &= \pi\left[\frac{3x^5}{5} - 6x^4 + 54x^2 + 81x\right]_{-1}^{3} \\ &= \pi\left[\frac{1944}{5} - \left(-\frac{168}{5}\right)\right] \\ &= \frac{2112\pi}{5} \text{ units}^3\end{aligned}$$

6.

$y = x^n$ and $y = x^{\frac{1}{n}}$ are symmetrical about $y = x$. Point of intersection is $(1, 1)$.

Since area of $A + C$ = area of B, we can subdivide B into 2 pieces B_1 and B_2, so $A = C = B_1 = B_2$, hence

$$\int_0^1 x^n \ \mathrm{d}x = \frac{1}{4}$$
$$\left[\frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{4}$$
$$\frac{1}{n+1} = \frac{1}{4}$$
$$n = 3$$