Assignment 22 solutions

1(a)

$$\int \frac{x}{(2x-1)^4}\ \mathrm{d}x = \int \frac{\frac{u+1}{2}}{u^4}\cdot\ \frac{1}{2}\mathrm{d}u$$

$$= \frac{1}{4}\int \frac{u+1}{u^4}\mathrm{d}u$$

$$= \frac{1}{4}\int \frac{1}{u^3}+\frac{1}{u^4}\mathrm{d}u$$

$$= \frac{1}{4}\left(-\frac{1}{2u^2}-\frac{1}{3u^3}\right)+c$$

$$= \frac{1}{4}\left(-\frac{1}{2(2x-1)^2}-\frac{1}{3(2x-1)^3}\right)+c$$

$$u = 2x-1$$

$$x = \frac{u+1}{2}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}$$

1(b)

$$\int \frac{u-1}{u+1}\cdot 2u\ \mathrm{d}u = \int \frac{2u^2-2u}{u+1}\ \mathrm{d}u$$

$$= \int \frac{2(u+1)^2-6(u+1)+4}{u+1}\ \mathrm{d}u$$

$$= \int 2(u+1)-6+\frac{4}{u+1}\ \mathrm{d}u$$

$$= \int 2u-4+\frac{4}{u+1}\ \mathrm{d}u$$

$$= u^2-4u+4\ln|u+1|+c$$

$$= x-4\sqrt{x}+4\ln\left(\sqrt{x}+1\right)+c$$

$$u = \sqrt{x}$$

$$x = u^2$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = 2u$$

1(c)

$$\int x^2\sin\left(x^3\right)\ \mathrm{d}x = \int u^{\frac{2}{3}}\sin u\ \cdot\frac{1}{3}u^{-\frac{2}{3}}\ \mathrm{d}u$$

$$= \int \frac{1}{3}\sin u\ \mathrm{d}u$$

$$= -\frac{1}{3}\cos u+c$$

$$= -\frac{1}{3}\cos\left(x^3\right)+c$$

$$u = x^3$$

$$x = u^{\frac{1}{3}}$$

$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{3}u^{-\frac{2}{3}}$$

1(d)

$$\int \frac{1}{x\ln\left(x^2\right)}\ \mathrm{d}x = \int \frac{1}{e^{\frac{1}{2}u}u}\cdot\frac{1}{2}e^{\frac{1}{2}u}\ \mathrm{d}u$$
$$= \int \frac{1}{2u}\ \mathrm{d}u$$
$$= \frac{1}{2}\ln|u| + c$$
$$= \frac{1}{2}\ln\left(\ln x^2\right) + c$$

$$u = \ln x^2$$
$$x^2 = e^u$$
$$x = e^{\frac{1}{2}u}$$
$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{2}e^{\frac{1}{2}u}$$

OR

$$\int \frac{1}{x\ln\left(x^2\right)}\ \mathrm{d}x = \int \frac{1}{x}\frac{1}{\ln\left(x^2\right)}\ \mathrm{d}x$$
$$= \frac{1}{2}\int \frac{2x}{x^2}\frac{1}{\ln\left(x^2\right)}\ \mathrm{d}x$$
$$= \frac{1}{2}\ln\left(\ln\left(x^2\right)\right) + c$$

1(e)

$$\int \mathrm{e}^{1+x+e^x}\ \mathrm{d}x = \int e^{1+\ln u+u}\cdot\frac{1}{u}\ \mathrm{d}u$$
$$= \int e\cdot u\cdot e^u\cdot\frac{1}{u}\ \mathrm{d}u$$
$$= \int e\cdot e^u\ \mathrm{d}u$$
$$= e\cdot e^u + c$$
$$= e\cdot e^{e^x} + c$$

$$u = e^x$$
$$x = \ln u$$
$$\frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{u}$$

OR

$$\int \mathrm{e}^{1+x+e^x}\ \mathrm{d}x = \int \mathrm{e}\cdot\mathrm{e}^x\cdot\mathrm{e}^{e^x}\ \mathrm{d}x$$
$$= e\int \mathrm{e}^x\cdot\mathrm{e}^{e^x}\ \mathrm{d}x$$
$$= e\cdot e^{e^x} + c$$
$$= e^{1+e^x} + c$$

1(f)

$$\int \left(\sin^{-1}x\right)^2\ \mathrm{d}x = \int u^2\cdot\cos u\ \mathrm{d}u$$
$$= u^2\left(\sin u\right) - \int \sin u\cdot 2u\ \mathrm{d}u$$
$$= u^2\sin u - 2\int u\sin u\ \mathrm{d}u$$
$$= u^2\sin u - 2\left\{u\left(-\cos u\right) - \int -\cos u\ \mathrm{d}u\right\}$$
$$= u^2\sin u + 2u\cos u - 2\sin u + c$$
$$= x\left(\sin^{-1}x\right)^2 + 2\sqrt{1-x^2}\ \sin^{-1}x - 2x + c$$

$$u = \sin^{-1}x$$
$$x = \sin u \rightarrow \cos u = \sqrt{1-x^2}$$
$$\frac{\mathrm{d}x}{\mathrm{d}u} = \cos u$$

2(a)

$$\int \left(x^2+x+1\right)\mathrm{e}^x\ \mathrm{d}x = \left(x^2+x+1\right)\mathrm{e}^x - \int \mathrm{e}^x\left(2x+1\right)\mathrm{d}x \quad \left[u = x^2+x+1\right]$$
$$= \left(x^2+x+1\right)\mathrm{e}^x - \left\{\left(2x+1\right)\mathrm{e}^x - \int \mathrm{e}^x\left(2\right)\mathrm{d}x\right\} \quad \left[u = 2x+1\right]$$
$$= \left(x^2+x+1\right)\mathrm{e}^x - \left(2x+1\right)\mathrm{e}^x + \int \mathrm{e}^x\left(2\right)\mathrm{d}x$$
$$= \left(x^2+x+1\right)\mathrm{e}^x - \left(2x+1\right)\mathrm{e}^x + 2\mathrm{e}^x + c$$

2(b)

$$\int x^3 \ln x \, \mathrm{d}x = \ln x\left(\frac{x^4}{4}\right) - \int \frac{x^4}{4}\left(\frac{1}{x}\right) \mathrm{d}x \quad [u = \ln x]$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 \, \mathrm{d}x$$

$$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + c$$

2(c)

$$\int x \sin\frac{x}{2} \mathrm{d}x = x\left(-2\cos\frac{x}{2}\right) - \int -2\cos\frac{x}{2} \, \mathrm{d}x \quad [u = x]$$

$$= -2x\cos\frac{x}{2} + 2\int \cos\frac{x}{2} \, \mathrm{d}x$$

$$= -2x\cos\frac{x}{2} + 4\sin\frac{x}{2} + c$$

2(d)

$$\int \ln(x+1) \, \mathrm{d}x = \ln(x+1)(x) - \int x\left(\frac{1}{x+1}\right) \mathrm{d}x \quad [u = \ln(x+1)]$$

$$= x\ln(x+1) - \int \frac{x+1-1}{x+1} \, \mathrm{d}x$$

$$= x\ln(x+1) - \int 1 - \frac{1}{x+1} \, \mathrm{d}x$$

$$= x\ln|x+1| - x + \ln|x+1| + c$$

2(e)

$$\int \mathrm{e}^{-x} \cos x \, \mathrm{d}x = \cos x \cdot -\mathrm{e}^{-x} - \int \mathrm{e}^{-x} \cdot \sin x \, \mathrm{d}x \quad [u = \cos x]$$

$$= -\mathrm{e}^{-x}\cos x - \left\{\sin x \cdot \left(-\mathrm{e}^{-x}\right) - \int -\mathrm{e}^{-x}\cos x \, \mathrm{d}x\right\} \quad [u = \sin x]$$

$$= -\mathrm{e}^{-x}\cos x + \mathrm{e}^{-x}\sin x - \int \mathrm{e}^{-x}\cos x \, \mathrm{d}x$$

$$\therefore 2\int \mathrm{e}^{-x}\cos x \, \mathrm{d}x = \mathrm{e}^{x}\sin x + \mathrm{e}^{-x}\sin x$$

$$\int \mathrm{e}^{-x}\cos x \, \mathrm{d}x = \frac{-\mathrm{e}^{-x}\cos x + \mathrm{e}^{-x}\sin x}{2} + c$$

2(f)

$$\int e^{2x}\cos 3x\ dx = \cos 3x\cdot\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cdot -3\sin 3x\ dx \quad [u=\cos 3x]$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\int e^{2x}\cdot\sin 3x\ dx \quad [u=\sin 3x]$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{2}\left\{\sin 3x\cdot\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\cdot 3\cos 3x\ dx\right\}$$

$$= \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x - \frac{9}{4}\int e^{2x}\cos 3x\ dx$$

$$\therefore \frac{13}{4}\int e^{2x}\cos 3x\ dx = \frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x$$

$$\int e^{2x}\cos 3x\ dx = \frac{4}{13}\left(\frac{1}{2}e^{2x}\cos 3x + \frac{3}{4}e^{2x}\sin 3x\right) + c$$

$$= \frac{4}{26}e^{2x}\cos 3x + \frac{12}{52}e^{2x}\sin 3x + c$$

2(g)

$$\int \sin(\ln x)\ dx = x\sin(\ln x) - \int \cos(\ln x)\ dx$$

$$= x\sin(\ln x) - \left\{x\cos(\ln x) - \int x\left(-\frac{1}{x}\sin(\ln x)\right)\right\} dx$$

$$= x\sin(\ln x) - x\cos(\ln x) - \int \sin(\ln x)\ dx$$

$$\therefore 2\int \sin(\ln x)\ dx = x\sin(\ln x) - x\cos(\ln x)$$

$$\int \sin(\ln x)\ dx = \frac{x\sin(\ln|x|) - x\cos(\ln|x|)}{2} + c$$

2(h)

$$\int x(\ln x)^2\ dx = (\ln x)^2\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{2}{x}\ln x\ dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \int x\ln x\ dx$$

$$= \frac{x^2}{2}(\ln x)^2 - \left\{(\ln x)\frac{x^2}{2} - \int \frac{x^2}{2}\cdot\frac{1}{x}\ dx\right\}$$

$$= \frac{x^2}{2}(\ln|x|)^2 - \frac{x^2}{2}(\ln|x|) + \frac{x^2}{4} + c$$