Year 4 Math Assignment 16: Partial Fractions Solutions

Q1 **(a)** $\dfrac{5}{(x+2)(x-2)} = \dfrac{A}{x+2} + \dfrac{B}{x-2}$.

By "cover-up" rule:

sub $x = -2 \to A = -\dfrac{5}{4}$

sub $x = 2 \to B = \dfrac{5}{4}$

Alternatively, by substitution:

$5 = A(x - 2) + B(x + 2)$

$x = 2 \;\Rightarrow 5 = B(2 + 2)$

$B = \dfrac{5}{4}$

$x = -2 \;\Rightarrow\; 5 = A(-2 - 2)$

$A = -\dfrac{5}{4}$

$$\frac{5}{(x+2)(x-2)} = -\frac{5}{4(x+2)} + \frac{5}{4(x-2)}.$$

(b) $\dfrac{5x+1}{2x^2+5x-3} = \dfrac{5x+1}{(2x-1)(x+3)} = \dfrac{A}{2x-1} + \dfrac{B}{x+3}$.

By "cover-up" rule:

sub $x = \dfrac{1}{2} \to A = \dfrac{\frac{5}{2}+1}{\frac{1}{2}+3} = 1$

sub $x = -3 \to B = \dfrac{-15+1}{-7} = 2$

Alternatively, by substitution:

$5x + 1 = A(x + 3) + B(2x - 1)$

$$\begin{aligned} x = -3 \;\Rightarrow\; 5(-3) + 1 &= B[2(-3) - 1] \\ -14 &= B(-7) \\ B &= 2 \end{aligned}$$

$$x = \frac{1}{2} \Rightarrow 5\left(\frac{1}{2}\right) + 1 = A\left(\frac{1}{2} + 3\right)$$

$$3\frac{1}{2} = A\left(3\frac{1}{2}\right)$$

$$A = 1$$

$$\frac{5x+1}{(2x-1)(x+3)} = \frac{1}{2x-1} + \frac{2}{x+3}.$$

(c) $$\frac{3x^2 - 9x - 1}{(x-1)(x-2)(x-4)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-4}$$

By “cover-up” rule:

sub $x = 1 \rightarrow A = -\frac{7}{3}$

sub $x = 3 \rightarrow B = \frac{7}{2}$

sub $x = 4 \rightarrow C = \frac{11}{6}$

Alternatively, by substitution:

$$3x^2 - 9x - 1 = A(x-2)(x-4) + B(x-1)(x-4) + C(x-1)(x-2)$$

$$x = 2 \Rightarrow -7 = -2B$$

$$B = \frac{7}{2}$$

$$x = 4 \Rightarrow 11 = 6C$$

$$C = \frac{11}{6}$$

$$x = 1 \Rightarrow -7 = 3A$$

$$A = -\frac{7}{3}$$

$$\frac{3x^2 - 9x - 1}{(x-1)(x-2)(x-4)} = -\frac{-7}{3(x-1)} + \frac{7}{2(x-2)} + \frac{11}{6(x-4)}$$

Q2 (a) $\dfrac{x^2+3x+1}{(x+1)(x+2)}$ → improper fraction

By long division, $\dfrac{x^2+3x+1}{(x+1)(x+2)} = 1 - \dfrac{1}{(x+1)(x+2)}$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

By "cover-up" rule:

sub $x = -1 \rightarrow A = 1$
sub $x = -2 \rightarrow B = -1$

Alternatively, by substitution:

$1 = A(x+2) + B(x+1)$

$x = -1 \;\Rightarrow\; 1 = A$

$x = -2 \;\Rightarrow\; 1 = -B$
$B = -1$

$$\frac{x^2+3x+1}{(x+1)(x+2)} = 1 - \frac{1}{x+1} + \frac{1}{x+2}$$

(b) $\dfrac{x^3-4x+5}{(x-2)(x+3)}$ → improper fraction

By long division, $\dfrac{x^3-4x+5}{(x-2)(x+3)} = x - 1 + \dfrac{3x-1}{(x-2)(x+3)}$

$$\frac{3x-1}{(x-2)(x+3)} = \frac{A}{x-2} + \frac{B}{x+3}$$

By "cover-up" rule:

sub $x = 2 \rightarrow A = 1$
sub $x = -3 \rightarrow B = 2$

Alternatively, by substitution:

$3x - 1 = A(x+3) + B(x-2)$

$x = -3 \Rightarrow -10 = -5B$

$B = 2$

$x = 2 \Rightarrow 5 = 5A$

$A = 1$

$$\frac{x^3 - 4x + 5}{(x-2)(x+3)} = x - 1 + \frac{1}{x-2} + \frac{2}{x+3}$$

(c) $$\frac{3x^2 - 2}{(x+1)(x^2+x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+x+1}$$

$$3x^2 - 2 = A(x^2+x+1) + (Bx+C)(x+1)$$

$x = -1 \Rightarrow A = 1$ [A can also be found using "cover-up" rule]

By comparing coefficients, $B = 2$, $C = -3$

$$\frac{3x^2 - 2}{(x+1)(x^2+x+1)} = \frac{1}{x+1} + \frac{2x-3}{x^2+x+1}$$

(d) $$\frac{4x^2 - 3x - 2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

$4x^2 - 3x - 2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$ [A and C can also be found using "cover-up" rule]

$x = 1 \Rightarrow 4(1)^2 - 3(1) - 2 = C(2)$

$C = -\frac{1}{2}$

$x = -1 \Rightarrow 4(-1)^2 - 3(-1) - 2 = A(4)$

$A = \frac{5}{4}$

Equating constant term: $-2 = A - B + C$

$-2 = \frac{5}{4} - B - \frac{1}{2}$

$B = \frac{11}{4}$

$$\frac{4x^2 - 3x - 2}{(x+1)(x-1)^2} = \frac{5}{4(x+1)} + \frac{11}{4(x-1)} - \frac{1}{2(x-1)^2}$$

(e) $\dfrac{4x-1}{x^2(x^2-4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2} + \dfrac{D}{x-2}$

$4x-1 = A(x)(x+2)(x-2) + B(x+2)(x-2) + C(x^2)(x-2) + D(x^2)(x+2)$

$x = -2 \Rightarrow -9 = C(4)(-4)$ [B, C and D can also be found using "cover-up" rule]

$C = \dfrac{9}{16}$

$x = 2 \Rightarrow 7 = D(4)(4)$

$D = \dfrac{7}{16}$

$x = 0 \Rightarrow -1 = B(1)(-4)$

$B = \dfrac{1}{4}$

Comparing coefficients of x^3, $A + C + D = 0 \rightarrow A = -1$

$\dfrac{4x-1}{x^2(x^2-4)} = -\dfrac{1}{x} + \dfrac{1}{4x^2} + \dfrac{9}{16(x+2)} + \dfrac{7}{16(x-2)}$

Q3 $2x^3 - 3x^2 - 8x + 12 = (x-2)(2x^2 + x - 6)$

$= (x-2)(2x-3)(x+2)$

$\dfrac{7}{(x-2)(2x-3)(x+2)} = \dfrac{A}{x-2} + \dfrac{B}{2x-3} + \dfrac{C}{x+2}$.

$7 = A(2x-3)(x+2) + B(x-2)(x+2) + C(x-2)(2x-3)$ Or use "cover-up" rule

$x = 2 \Rightarrow 7 = A(1)(4)$

$A = \dfrac{7}{4}$

$x = -2 \Rightarrow 7 = C(-4)(-7)$

$C = \dfrac{1}{4}$

$x = \dfrac{3}{2} \Rightarrow 7 = B\left(-\dfrac{1}{2}\right)\left(\dfrac{7}{2}\right)$

$B = -4$

Thus, $\dfrac{7}{(x-2)(2x-3)(x+2)} = \dfrac{7}{4(x-2)} - \dfrac{4}{2x-3} + \dfrac{1}{4(x+2)}$.

Q4 $\dfrac{x^3}{x^6-1}=\dfrac{x^3}{(x^3-1)(x^3+1)}$

Let $y=x^3$

$$\frac{x^3}{(x^3-1)(x^3+1)}=\frac{y}{(y-1)(y+1)}$$

$$\frac{y}{(y-1)(y+1)}=\frac{A}{y-1}+\frac{B}{y+1}$$

By "cover-up" rule, $A=\dfrac{1}{2}$, $B=\dfrac{1}{2}$

$$\frac{x^3}{(x^3-1)(x^3+1)}=\frac{1}{2(x^3-1)}+\frac{1}{2(x^3+1)}$$

$$\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$$

$$=\frac{C}{x-1}+\frac{Dx+E}{x^2+x+1}$$

By "cover-up" rule, $C=\dfrac{1}{3}$

$$1=(Dx+E)(x-1)+\frac{1}{3}(x^2+x+1)$$

By comparing coefficients,

$$\Rightarrow D=-\frac{1}{3}, E=-\frac{2}{3}$$

$$\frac{1}{x^3-1}=\frac{1}{(x-1)(x^2+x+1)}$$

$$=\frac{1}{3(x-1)}-\frac{x+2}{3(x^2+x+1)}$$

Replacing "x" by "$-x$":

$$\frac{1}{-x^3-1}=\frac{1}{3(-x-1)}-\frac{-x+2}{3(x^2-x+1)}$$

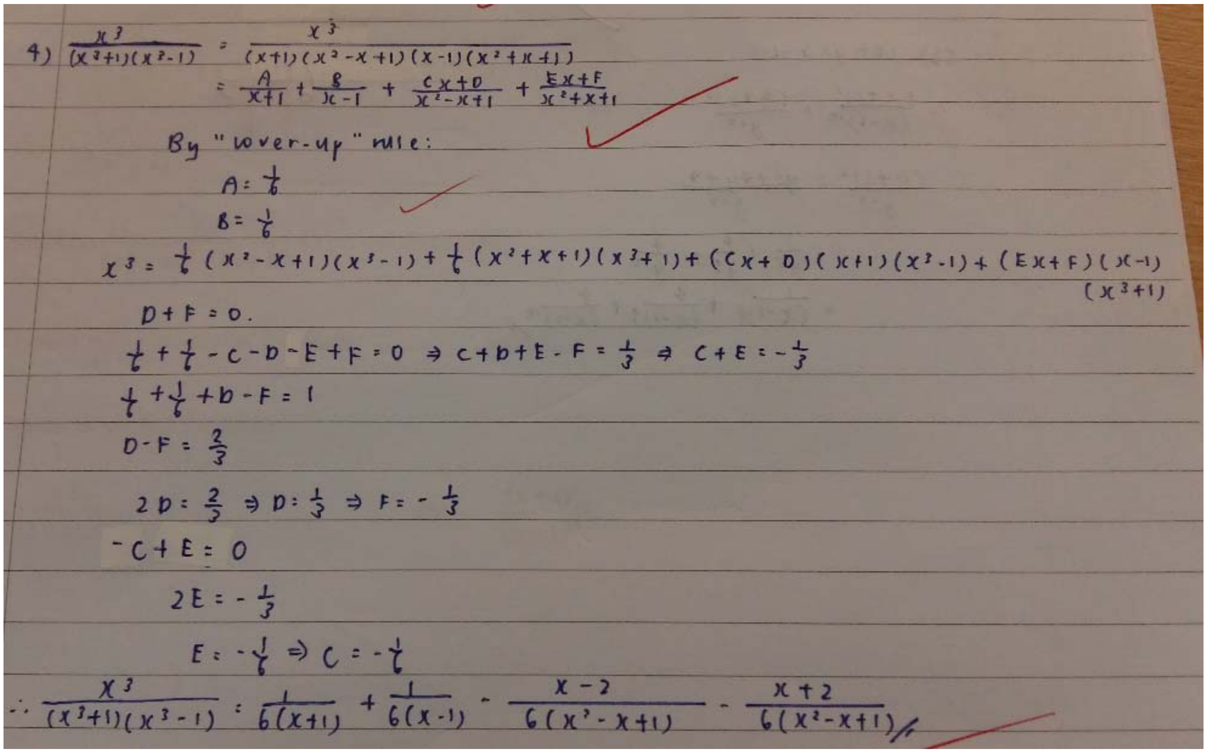
$$-\frac{1}{(x^3+1)}=-\frac{1}{3(x+1)}+\frac{x-2}{3(x^2-x+1)}$$

$$\frac{1}{(x^3+1)}=\frac{1}{3(x+1)}-\frac{x-2}{3(x^2-x+1)}$$

Therefore, $\dfrac{x^3}{(x^3-1)(x^3+1)}=\dfrac{1}{2}\left(\dfrac{1}{3(x-1)}-\dfrac{x+2}{3(x^2+x+1)}\right)+\dfrac{1}{2}\left(\dfrac{1}{3(x+1)}-\dfrac{x-2}{3(x^2-x+1)}\right)$

$$=\frac{1}{6(x-1)}-\frac{x+2}{6(x^2+x+1)}+\frac{1}{6(x+1)}-\frac{x-2}{6(x^2-x+1)}$$

Alternative Q4 (contributed by 20154S3 Kwang Haoyang)



4) $\frac{x^3}{(x^3+1)(x^3-1)} = \frac{x^3}{(x+1)(x^2-x+1)(x-1)(x^2+x+1)}$

$= \frac{A}{x+1} + \frac{B}{x-1} + \frac{Cx+D}{x^2-x+1} + \frac{Ex+F}{x^2+x+1}$

By "cover-up" rule:

$A = \frac{1}{6}$

$B = \frac{1}{6}$

$x^3 = \frac{1}{6}(x^2-x+1)(x^3-1) + \frac{1}{6}(x^2+x+1)(x^3+1) + (Cx+D)(x+1)(x^3-1) + (Ex+F)(x-1)(x^3+1)$

$D + F = 0.$

$\frac{1}{6} + \frac{1}{6} - C - D - E + F = 0 \Rightarrow C + D + E - F = \frac{1}{3} \Rightarrow C + E = -\frac{1}{3}$

$\frac{1}{6} + \frac{1}{6} + D - F = 1$

$D - F = \frac{2}{3}$

$2D = \frac{2}{3} \Rightarrow D = \frac{1}{3} \Rightarrow F = -\frac{1}{3}$

$-C + E = 0$

$2E = -\frac{1}{3}$

$E = -\frac{1}{6} \Rightarrow C = -\frac{1}{6}$

$\therefore \frac{x^3}{(x^3+1)(x^3-1)} = \frac{1}{6(x+1)} + \frac{1}{6(x-1)} - \frac{x-2}{6(x^2-x+1)} - \frac{x+2}{6(x^2-x+1)}$

Minor error (should be x^2+x+1 for one of the denominators)

Q5(a)

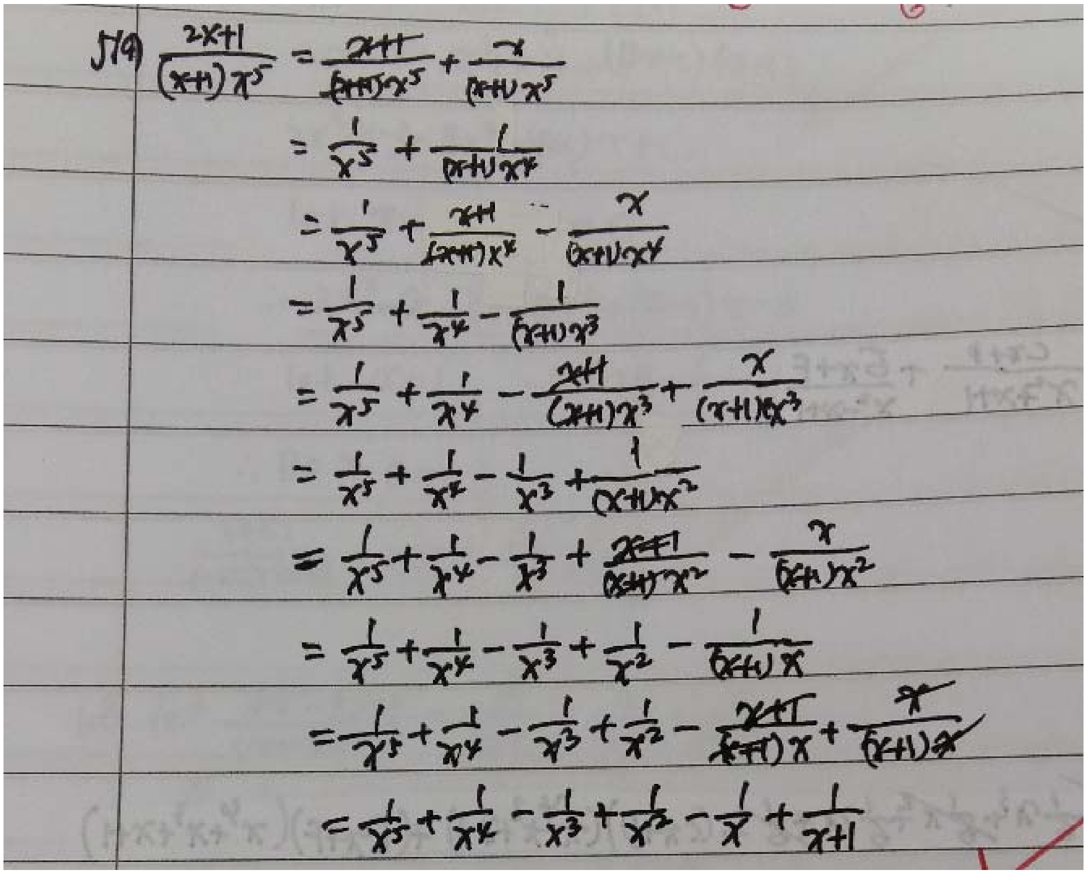
$$\frac{2x+1}{(x+1)x^5} = \left(\frac{1+2x}{1+x}\right)\left(\frac{1}{x^5}\right)$$

$$\begin{array}{r l}
 & 1 + x - x^2 + x^3 - x^4 \\
1+x \,\big) & 1 + 2x \\
 & 1 + x \\
\hline
 & x \\
 & x + x^2 \\
\hline
 & -x^2 \\
 & -x^2 - x^3 \\
\hline
 & x^3 \\
 & x^3 + x^4 \\
\hline
 & -x^4 \\
 & -x^4 - x^5 \\
\hline
 & x^5
\end{array}$$

$$\frac{2x+1}{(x+1)x^5}=\left(1+x-x^2+x^3-x^4+\frac{x^5}{x+1}\right)\left(\frac{1}{x^5}\right)$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x+1}$$

M2: Contributed by 20154S2 Wang Shuwei



5(a) $$\frac{2x+1}{(x+1)x^5}=\frac{x+1}{(x+1)x^5}+\frac{x}{(x+1)x^5}$$

$$=\frac{1}{x^5}+\frac{1}{(x+1)x^4}$$

$$=\frac{1}{x^5}+\frac{x+1}{(x+1)x^4}-\frac{x}{(x+1)x^4}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{(x+1)x^3}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{x+1}{(x+1)x^3}+\frac{x}{(x+1)x^3}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{1}{(x+1)x^2}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{x+1}{(x+1)x^2}-\frac{x}{(x+1)x^2}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{(x+1)x}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{1}{x^2}-\frac{x+1}{(x+1)x}+\frac{x}{(x+1)x}$$

$$=\frac{1}{x^5}+\frac{1}{x^4}-\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}+\frac{1}{x+1}$$

Q5(b)

M1:

$$\frac{(x+1)^2}{(x-1)^{10}}=\frac{A_1}{(x-1)}+\frac{A_2}{(x-1)^2}+\frac{A_3}{(x-1)^3}+\frac{A_4}{(x-1)^4}+\frac{A_5}{(x-1)^5}+\frac{A_6}{(x-1)^6}+\frac{A_7}{(x-1)^7}+\frac{A_8}{(x-1)^8}+\frac{A_9}{(x-1)^9}+\frac{A_{10}}{(x-1)^{10}}$$

$$(x+1)^2=A_1(x-1)^9+A_2(x-1)^8+A_3(x-1)^7+A_4(x-1)^6+A_5(x-1)^5+A_6(x-1)^4+A_7(x-1)^3+A_8(x-1)^2+A_9(x-1)+A_{10}$$

Comparing coefficients of:

$x^9,x^8,x^7,x^6,x^5,x^4,x^3\rightarrow A_1,A_2,A_3,A_4,A_5,A_6,A_7=0$

$\therefore x^2+2x+1=A_8(x-1)^2+A_9(x-1)+A_{10}$

Comparing coefficients of:

$x^2\rightarrow 1=A_8$

$x \to 2 = -2A_8 + A_9$

$A_9 = 4$

constant $\to 1 = A_8 - A_9 + A_{10}$

$A_{10} = 4$

$$\therefore \frac{(x+1)^2}{(x-1)^{10}} = \frac{1}{(x-1)^8} + \frac{4}{(x-1)^9} + \frac{4}{(x-1)^{10}}$$

M2:

Let $y = x - 1$

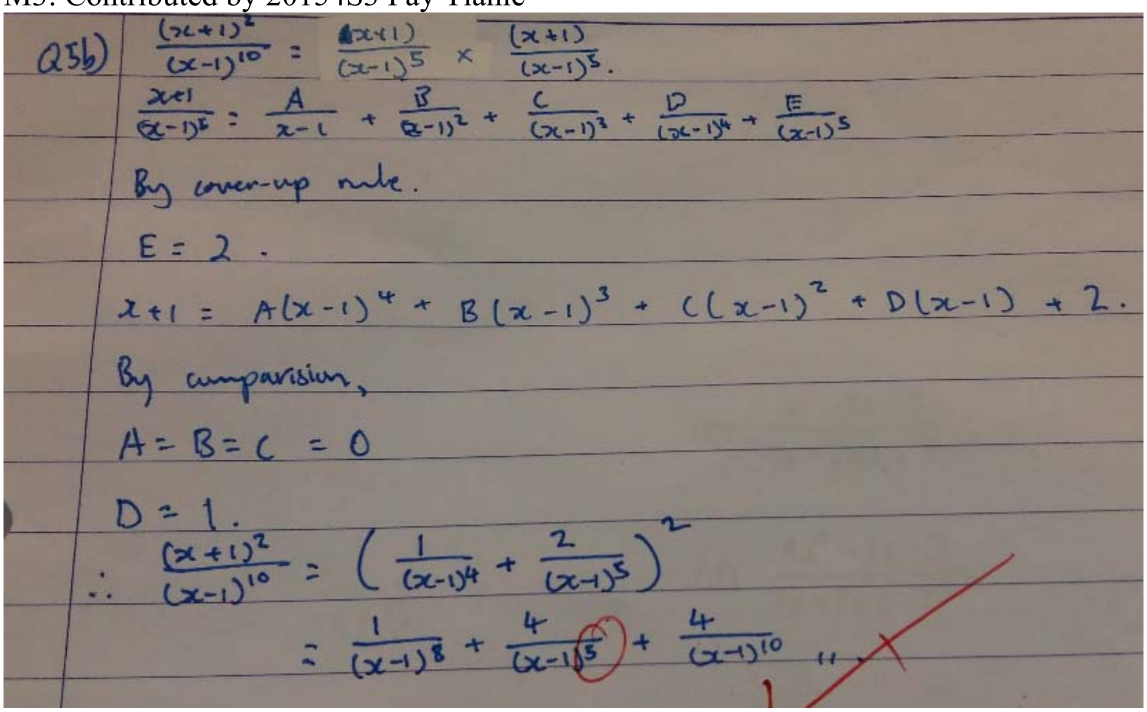
$$\therefore \frac{(x+1)^2}{(x-1)^{10}} = \frac{(y+2)^2}{y^{10}}$$

$$\frac{(y+2)^2}{y^{10}} = \frac{y^2+4y+4}{y^{10}}$$

$$= \frac{1}{y^8} + \frac{4}{y^9} + \frac{4}{y^{10}}$$

$$= \frac{1}{(x-1)^8} + \frac{4}{(x-1)^9} + \frac{4}{(x-1)^{10}}$$

M3: Contributed by 20154S3 Pay Tianle



Minor error in power (should be 9)

M4: Contributed by 20154S3 William Kin

$$\frac{(x+1)^2}{(x-1)^{10}} = \left(\frac{x+1}{x-1}\right)^2 \frac{1}{(x-1)^8}$$

$$= \left(\frac{x-1+2}{x-1}\right)^2 \frac{1}{(x-1)^8}$$

$$= \left(1+\frac{2}{x-1}\right)^2 \frac{1}{(x-1)^8}$$

$$= \left(1+\frac{4}{x-1}+\frac{4}{(x-1)^2}\right)\frac{1}{(x-1)^8}$$

$$= \frac{1}{(x-1)^8}+\frac{4}{(x-1)^9}+\frac{4}{(x-1)^{10}}$$