<u>Year 4 Math Assignment 11: Stationary Points solutions</u>

1 $y = 3x^2 - \frac{1}{x} - 7x + 3$

$\frac{dy}{dx} = 6x + \frac{1}{x^2} - 7$

$\frac{d^2y}{dx^2} = 6 - \frac{2}{x^3}$

At stationary points, $\frac{dy}{dx} = 0 \Rightarrow 6x + \frac{1}{x^2} - 7 = 0$

$6x^3 + 1 - 7x^2 = 0$

$(x - 1)(6x^2 - x - 1) = 0$

$(x - 1)(3x + 1)(2x - 1) = 0$

$x = 1$ or $x = -\frac{1}{3}$ or $x = \frac{1}{2}$

When $x = 1$, $y = -2$, $\frac{d^2y}{dx^2} > 0 \Rightarrow (1, -2)$ is a minimum point

When $x = -\frac{1}{3}$, $y = 8\frac{2}{3}$, $\frac{d^2y}{dx^2} > 0 \Rightarrow \left(-\frac{1}{3}, 8\frac{2}{3}\right)$ is a minimum point

When $x = \frac{1}{2}$, $y = -1\frac{3}{4}$ $\frac{d^2y}{dx^2} < 0 \Rightarrow \left(\frac{1}{2}, -1\frac{3}{4}\right)$ is a maximum point

2 (a) Perimeter of square = $4x$
Circumference of circle = $2\pi r$

$24 = 4x + 2\pi r$

$r = \frac{24 - 4x}{2\pi}$

$= \frac{12 - 2x}{\pi}$

(b) Area of square = x^2
Area of circle = πr^2

Let A be the sum of the areas of the square and the circle.

$$A = x^2 + \pi r^2$$

$$= x^2 + \pi\left(\frac{12-2x}{\pi}\right)^2$$

$$= x^2 + \frac{(12-2x)^2}{\pi}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2x + \frac{2}{\pi}(12-2x)(-2)$$

$$= 2x - \frac{48}{\pi} + \frac{8x}{\pi}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 2 + \frac{8}{\pi}$$

At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 2x - \frac{48}{\pi} + \frac{8x}{\pi} = 0$

$$\left(2 + \frac{8}{\pi}\right)x = \frac{48}{\pi}$$

$$x = \frac{48}{\pi}\left(\frac{\pi}{2\pi + 8}\right)$$

$$= \frac{24}{\pi + 4}$$

At $x = \frac{24}{\pi+4}$, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} > 0 \Rightarrow x = \frac{24}{\pi+4}$ is a minimum point.

3 (a) $16x + 2(2r) + \pi r = 560$

$$x = 35 - \frac{\pi + 4}{16}r$$

A = Area of triangle BCQ + area of rectangle $ABCD$ + area of semicircle

$$= \frac{1}{2}(2r)\left(\sqrt{3}r\right) + 8x(2r) + \frac{1}{2}\pi r^2$$

$$= \sqrt{3}r^2 + 16r\left(35 - \frac{\pi+4}{16}r\right) + \frac{\pi r^2}{2}$$

$$= 560r + \sqrt{3}r^2 - (\pi + 4)r^2 + \frac{\pi r^2}{2}$$

$$= 560r + \left(\sqrt{3} - 4 - \frac{\pi}{2}\right)r^2 \quad \text{(shown)}$$

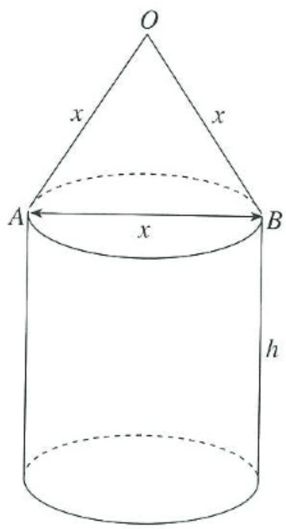
(b) $\frac{\mathrm{d}A}{\mathrm{d}r} = 560 + \left(\sqrt{3} - 4 - \frac{\pi}{2}\right)r$

At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}r} = 0 \Rightarrow 560 + 2\left(\sqrt{3} - 4 - \frac{\pi}{2}\right)r = 0$

$r = 72.9$

$\frac{\mathrm{d}^2 A}{\mathrm{d}r^2} = 2\sqrt{3} - 4 - \frac{\pi}{2} < 0 \Rightarrow A$ is a maximum when $r = 72.9$

4



$\Delta\, OAB$ is an isosceles triangle, thus, each of its angles is 60°.

sin 60° = $\frac{\text{height of cone}}{x} \Leftrightarrow$ height of cone = $\frac{\sqrt{3}}{2}x$

Volume of toy = 60π

$$\pi\left(\frac{x}{2}\right)^2 h + \frac{1}{3}\pi\left(\frac{x}{2}\right)^2 \frac{\sqrt{3}}{2}x = 60\pi$$

$$\frac{\pi x^2 h}{4} + \frac{\sqrt{3}\pi x^3}{24} = 60\pi$$

$$\frac{\pi x^2 h}{4} = 60\pi - \frac{\sqrt{3}\pi x^3}{24}$$

$$h = \frac{240}{x^2} - \frac{\sqrt{3}}{6}x$$

Total surface area of the toy, $A = \pi rl + 2\pi rh + \pi r^2$

$$= \pi\left(\frac{x}{2}\right)x + 2\pi\left(\frac{x}{2}\right)h + \pi\left(\frac{x}{2}\right)^2$$

$$= \frac{3\pi x^2}{4} + \pi xh$$

$$= \frac{3\pi x^2}{4} + \pi x\left(\frac{240}{x^2} - \frac{\sqrt{3}}{6}x\right)$$

$$= \frac{3\pi x^2}{4} + \frac{240\pi}{x} - \frac{\sqrt{3}\pi x^2}{6}$$

$$= \pi x^2\left(\frac{3}{4} - \frac{\sqrt{3}}{6}\right) + \frac{240\pi}{x} \quad \text{(shown)}$$

$$\frac{\mathrm{d}A}{\mathrm{d}x} = 2\pi x\left(\frac{3}{4} - \frac{\sqrt{3}}{6}\right) - \frac{240\pi}{x^2}$$

$$\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = 2\pi\left(\frac{3}{4} - \frac{\sqrt{3}}{6}\right) + \frac{480\pi}{x^3}$$

At stationary point, $\frac{\mathrm{d}A}{\mathrm{d}x} = 0 \Rightarrow 2\pi x\left(\frac{3}{4} - \frac{\sqrt{3}}{6}\right) - \frac{240\pi}{x^2} = 0$

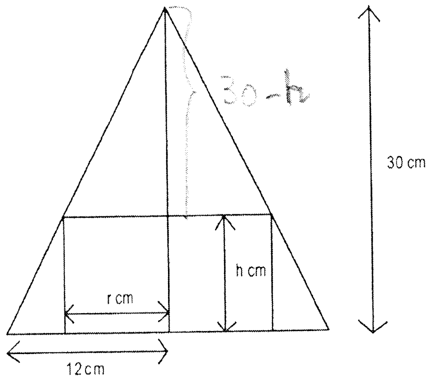
$$\frac{240\pi}{x^2} = 2\pi x\left(\frac{3}{4} - \frac{\sqrt{3}}{6}\right)$$

$$x^3 = \frac{120}{\frac{3}{4} - \frac{\sqrt{3}}{6}}$$

$$x = 6.38 \quad \text{(to 3 sig. fig)}$$

When $x = 6.38$ cm, $\frac{\mathrm{d}^2A}{\mathrm{d}x^2} > 0 \Rightarrow A$ is a minimum when $x = 6.38$.

5 The diagram shows the cross-section of a hollow cone of height 30 cm and base radius 12 cm and a solid cylinder of radius r cm and height h cm. Both stand on a horizontal surface with the cylinder inside the cone. The upper circular edge of the cylinder is in constant with the cone.



(i) Express h in terms of r and hence show that the volume, $V\ cm^3$, of the cylinder is given by

$$V = \pi(30r^2 - \frac{5}{2}r^3)$$

Given that r can vary,

(ii) find the volume of the largest cylinder which can stand inside the cone and show that, in this case, the cylinder occupies $\frac{4}{9}$ of the volume of the cone.

(i) By similar Δs, $\frac{30-h}{30} = \frac{r}{12}$

$$360 - 12h = 30r$$

$$h = \frac{360 - 30r}{12}$$

$$h = 30 - \frac{5}{2}r \quad \#$$

Vol of cylinder $= \pi r^2 h$

$$= \pi r^2 (30 - \frac{5}{2}r)$$

$$V = \pi(30r^2 - \frac{5}{2}r^3) \ \# \text{ shown.}$$

(ii) $\frac{dV}{dr} = \pi(60r - \frac{15}{2}r^2) = 0$

$$60r = \frac{15}{2}r^2$$

$$15r^2 - 120r = 0$$

$$r(15r - 120) = 0$$

$$r = 0 \text{ (rej) or } r = 8 \text{ cm.}$$

When $r = 8$, $V = \pi(30(8)^2 - \frac{5}{2}(8)^3)$

$$= 640\pi.$$

$$\frac{640\pi}{\frac{1}{3}\pi(12)^2(30)} = \frac{4}{9} \ \# \text{ shown.}$$

Check: $\frac{d^2V}{dr^2} = \pi(60 - 15r)$

When $r = 8$, $\frac{d^2V}{dr^2} < 0 \Rightarrow$ max volume.

Q6) $x+y=10$

$y = 10-x$

$x^3y^2 = x^3(10-x)^2$

$= x^3(100-20x+x^2)$

$= 100x^3 - 20x^4 + x^5$

$\frac{d(x^3y^2)}{dx} = 300x^2 - 80x^3 + 5x^4 = 0$

$5x^2(x^2 - 16x + 60) = 0$

$x^2 = 0$ or $x = 10$ or 6

$\frac{d^2(x^3y^2)}{dx^2} = 600x - 240x^2 + 20x^3$

when $x=10$, $y=0$ $\Rightarrow$ $(10,0)$ is min pt. $\left(\frac{d^2(x^3y^2)}{dx^2} > 0\right)$

when $x=6$, $y=4$ $\Rightarrow$ $(6,4)$ is max pt. $\left(\frac{d^2(x^3y^2)}{dx^2} < 0\right)$

when $x=0$, $y=10$ $\Rightarrow$ $(0,10)$ is pt. of inflexion.

x	0^-	0	0^+
$\frac{d(x^3y^2)}{dx}$	>0	0	>0
tangent	/	—	/

$\therefore$ max value of $x^3y^2 = 6^3 \cdot 4^2$

$= 3456$ #

min value of $x^3y^2 = 10^3 \cdot 0^2$

$= 0$ #

However, since $x>0$, $y>0$ $\Rightarrow$ only the max. value is accepted.