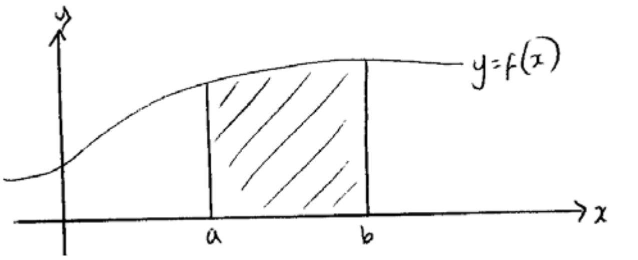
Finding Areas Under Curves

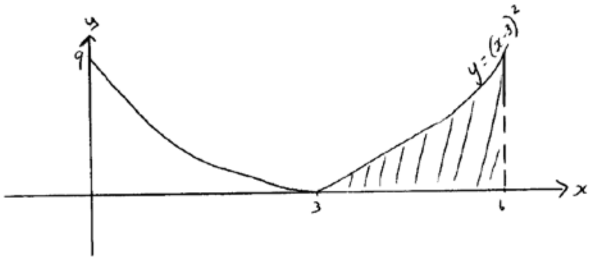
To find the area between a curve, the x-axis and the lines $x = a$ and $x = b$,

$$\text{Area} = \int_b^a y \ dx$$

where $y = f(x)$ is the equation of the curve



Example 1: Find the area under the curve $y = (x - 3)^2$ between $x = 3$ and $x = 6$.



$$= \int_3^6 (x - 3)^2 \ dx$$

$$= \int_3^6 x^2 - 6x + 9 \ dx$$

$$= \left[\frac{x^3}{3} - 3x^2 + 9x \right]_3^6$$

$$= \left(\frac{216}{3} - 3 \times 6^2 + 9 \times 6 \right) - \left(\frac{27}{3} - 3 \times 3^2 + 9 \times 3 \right)$$

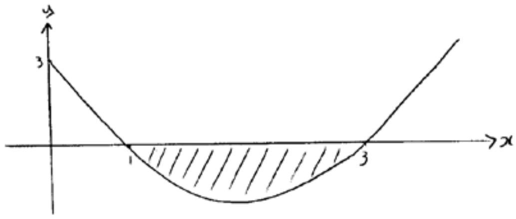
$= (72 + 108 + 54) - (9 - 27 + 27)$

$= 18 - 9$

$= 9$

Finding Area of Curves under the *x*-axis

Example 2: Find the area of the region bounded by the curve $y = x^2 - 4x + 3$ and the x-axis.



$f(x) = x^2 - 4x + 3$

$y = 0 \quad (x - 1)(x - 3) = 0$

$x = 1$ or $x = 3$ this means it cuts the x axis at 1 and 3

$$\int_1^3 x^2 - 4x + 3 \ dx$$

$$\left[\frac{x^3}{3} - 2x^2 + 3x \right]_1^3$$

$$= \left(\frac{3^3}{3} - 2 \times 3^2 + 3 \times 3 \right) - \left(\frac{1^3}{3} - 2 \times 1^2 + 3 \times 1 \right)$$

$$= (9 - 18 + 9) - \left(\frac{1}{3} - 2 + 3 \right)$$

$$= (0) - \left(1\frac{1}{3} \right)$$

$$= -1\frac{1}{3}$$

$\therefore$ Area $= 1\frac{1}{3}$ An area cannot be negative so ignore the sign

**Take the absolute value (modulus) of the Area if the answer is negative. Do this before adding separate areas together.

** If the areas you are required to find – one part is above the *x*-axis and the other is below, you will need to split them up into 2 pieces then add the areas together.

Example 3: Find the area of the region bounded by the curve $y = x^3 - x^2 - 6x$ and the x- axis.

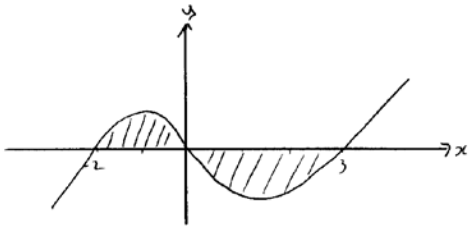
$$y = x^3 - x^2 - 6x$$

$$y = 0 \qquad 0 = x^3 - x^2 - 6x$$

$$0 = x(x^2 - x - 6)$$

$$0 = x(x + 2)(x - 3)$$

$$x = 0 \quad x = -2 \quad x = 3$$



Because there are two areas, deal with them separately

$$\int_{-2}^{0} x^3 - x^2 - 6x \; dx \qquad + \qquad \int_{0}^{3} x^3 - x^2 - 6x \; dx$$

$$= \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x \right]_{-2}^{0} \qquad = \left[\frac{x^4}{4} - \frac{x^3}{3} - 3x \right]_{0}^{3}$$

$$= (0) - \left(4 - \frac{4}{32} + 6 \right) \qquad = \left(\frac{81}{4} - 9 - 9 \right) - (0)$$

$$= -8\frac{2}{3} \qquad = 2\frac{1}{4}$$

$$\therefore \text{ Total area } = 8\frac{2}{3} + 2\frac{1}{4}$$

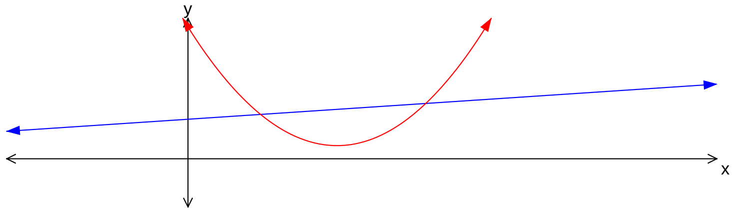
$$= 10\frac{11}{12}$$

<u>Finding the Area between a Curve and a Line</u>

The area between a line (equation y_1) and a curve (equation y_2) is given by

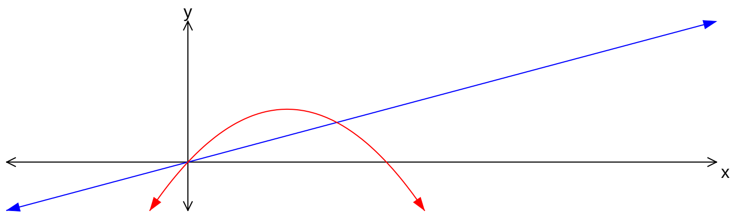
$$\text{Area} = \int_a^b (y_1 - y_2) \; dx$$

Why?



If you consider the area under the straight line (y_1) and subtract the area under the curve (y_2) then you would be left with the area in-between them.

Example 4: Find the area of the region bounded by the curve $y = x(4 - x)$ and the line $y = x$.



$$x = x(4 - x)$$

$$x = 4x - x^2$$

$$x^2 - 3x = 0$$

$$x(x - 3) = 0$$

$\therefore \; x = 0$ and $x = 3$ this means the points of intersection are 0 and 3

$$= \int_0^3 (y_2 - y_1) \; dx \qquad y_2 - y_1 = x - x(4 - x) = x^2 - 3x$$

$$= \int_0^3 x^2 - 3x \; dx$$

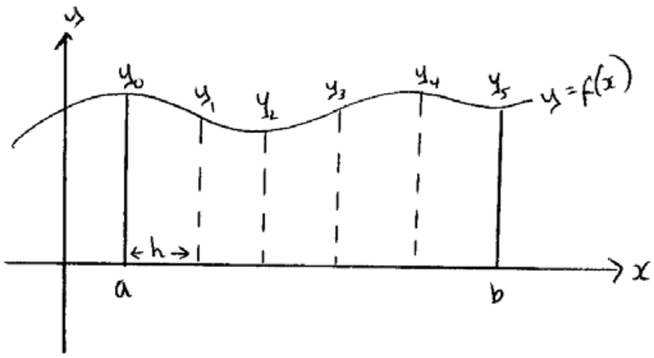
$$= \left[\frac{x^3}{3} - \frac{3x^2}{2} \right]_0^3$$

$$= \left(\frac{3^3}{3} - \frac{3 \times 3^2}{2} \right) - (0)$$

$$= -4{\cdot}5 \qquad \therefore \; \text{Area} = 4{\cdot}5$$

Trapezoidal Rule

If you want to find the area under a curve but you cannot integrate then you can use the trapezoidal rule.



Normally to find the area we would $\int_a^b y \; dx$. Instead, now we divide the area into equal strips. Then, find the area of each by approximating their area to that of trapeziums.

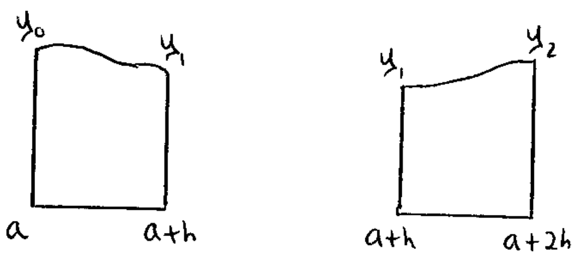
$$\text{Triangle} = \frac{1}{2} \times \text{base} \times \text{perpendicular height}$$

$$\text{Trapezium} = \frac{\text{sum of parallel sides}}{2} \times \text{height}$$

Step 1: We decide what the width of each trapezium (h) will be by deciding how many strips we are going to have then using this formula

$$h = \frac{b - a}{n} \qquad \text{where n is the number of strips}$$

Step 2: now we have the x values so we find the corresponding y values by substituting it into the original equations, these tell us the heights of the trapeziums.



Step 3: Use the formula to find the area.

$$\int_a^b y \; dx = \frac{1}{2} h \left[y_0 + 2(y_1 + y_2 + \ldots\ldots y_{n-1}) + y_n \right]$$

where $h = \dfrac{b - a}{n}$

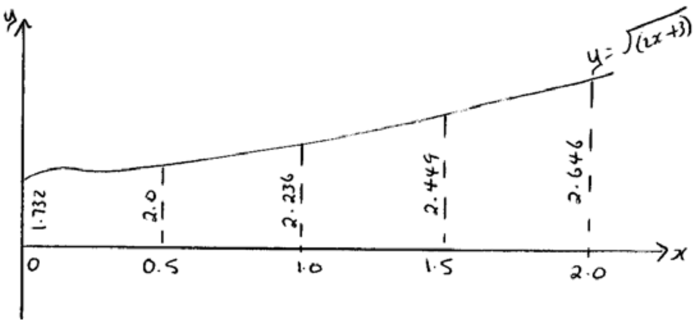
Example 5: Use the trapezium rule with 4 strips to estimate the area under the curve $y = \sqrt{2x + 3}$ between the lines $x = 0$ and $x = 2$

$$\text{strip width} = \frac{b - a}{n} \quad \text{where a} = 0, \text{ b} = 2 \text{ and n} = 4$$

$$h = \frac{2 - 0}{4}$$

$$h = 0{\cdot}5$$

x	0	0.5	1	1.5	2
y	1.732	2	2.236	2.449	2.646



$$Area = \frac{1}{2} \times 0{\cdot}5 \times [1{\cdot}732 \times 2(2 + 2{\cdot}236 + 2{\cdot}449) + 2{\cdot}646\}$$

$$A = \frac{1}{2} \times 0{\cdot}5 \times 17{\cdot}748$$

$$A = 4{\cdot}437$$

Remember, this is just an estimate of the area. If we know Integration, we can get the exact area. Increasing the number of strips will lead to a more accurate area.

**The Trapezium Rule is just a method to estimate the area if you do not know the equation of the curve. It is generally NOT used.