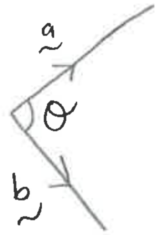
Example 18: Find the angle between the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

Solution:



$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\end{pmatrix}}{\left(\sqrt{3^2+1^2}\right)\left(\sqrt{2^2+(-3)^2}\right)}$$

$$= \frac{6-3}{\sqrt{10}\sqrt{13}}$$

$$= \frac{3}{\sqrt{130}}$$

$$\theta = 74.7°$$

Example 19:

(a) If $\underset{\sim}{a} \perp \underset{\sim}{b}$, show that $|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$

(b) Given that $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, and $|\underset{\sim}{b}| = \sqrt{2}|\underset{\sim}{a}|$, show that $2\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

Solution:

(a)

$$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$\therefore |\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$$

(b)

$$(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a} = 0 \quad \text{since } \underset{\sim}{a}+\underset{\sim}{b}\perp\underset{\sim}{a}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{a} = 0$$

$$\underset{\sim}{b}\cdot\underset{\sim}{a} = -|\underset{\sim}{a}|^2$$

$$(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{b} = 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$$

$$= -2|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

$$= -2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}|^2 \quad \left[\text{Given } |\underset{\sim}{b}| = \sqrt{2}|\underset{\sim}{a}|\right]$$

$$= 0$$

$$\therefore 2\underset{\sim}{a}+\underset{\sim}{b}\perp\underset{\sim}{b}$$

Example 20: If $3\underset{\sim}{i}+\lambda\underset{\sim}{j}$ and $2\lambda^2\underset{\sim}{i}-\underset{\sim}{j}$ are perpendicular vectors, find the value(s) of λ.

$\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

Solution:

$$(3\underset{\sim}{i}+\lambda\underset{\sim}{j})\cdot(2\lambda^2\underset{\sim}{i}-\underset{\sim}{j}) = 0$$

$$\begin{pmatrix}3\\\lambda\end{pmatrix}\cdot\begin{pmatrix}2\lambda^2\\-1\end{pmatrix} = 0$$

$$6\lambda^2 - \lambda = 0$$

$$\lambda = 0 \text{ or } \frac{1}{6}$$