Example 1

State whether the following statement is true or false.

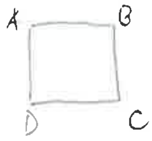
(i) $|\overrightarrow{AB}| = |\overrightarrow{CD}| \Rightarrow \overrightarrow{AB} = \overrightarrow{CD}$, (ii) $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow |\overrightarrow{AB}| = |\overrightarrow{CD}|$.

Solution: (i) False, though AB has the same magnitude as CD, they may not have the same direction.

(ii) True, if two vectors are equal, then both their magnitude and their direction must be the same.

Example 2

Given that $ABCD$ is a square,



(i) is $\overrightarrow{AB} = \overrightarrow{BC}$? (ii) is $\overrightarrow{AD} = \overrightarrow{BC}$? (iii) is $|\overrightarrow{AB}| = |\overrightarrow{BC}|$?

Solution: (i) No, different directions (ii) Yes (iii) Yes

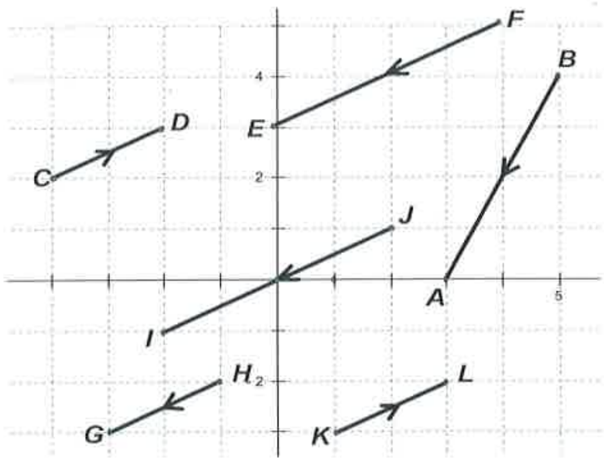
Example 3

State whether the following statement is true or false.

(i) $\overrightarrow{AB} = \overrightarrow{EF}$,

(ii) $|\overrightarrow{AB}| = |\overrightarrow{EF}| = |\overrightarrow{IJ}|$,

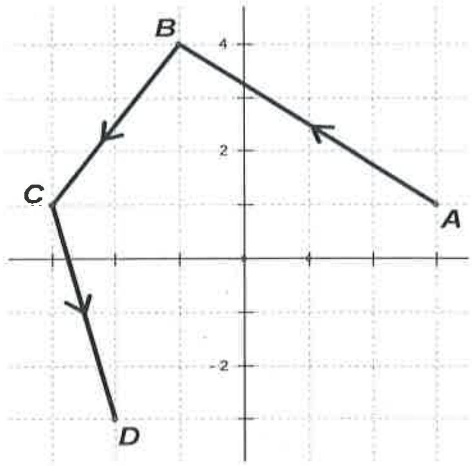
(iii) $\overrightarrow{CD} = \overrightarrow{KL} = -\overrightarrow{HG}$.



Solution: (i) False (ii) True (iii) True

Example 4

A boy kicked the ball from A to B, B to C and then from C to D.



(a) Express each of the following as a column vector.

(i) $\overrightarrow{AB}$, (ii) $\overrightarrow{BC}$, (iii) $\overrightarrow{CD}$, (iv) $\overrightarrow{AD}$.

(b) State whether the following statement is true or false.

(i) The sum of the magnitude $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}|$ represents the total distance traveled by the ball.

(ii) The vector $\overrightarrow{AD}$ represents the displacement of the ball.

(iii) $|\overrightarrow{AD}| = |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}|$?

(iv) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$?

Solution:

(a) (i) $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, (ii) $\overrightarrow{BC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, (iii) $\overrightarrow{CD} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, (iv) $\overrightarrow{AD} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

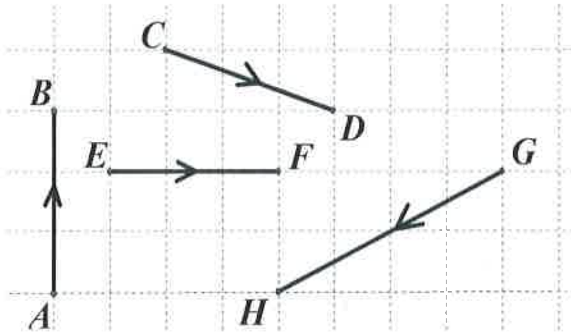
(b) (i) True, (ii) True,

(iii) False, because $|\overrightarrow{AD}| = \sqrt{5^2 + 4^2} = \sqrt{41}$, $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| = 5 + \sqrt{13} + \sqrt{17}$

(iv) True $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

Example 5

If vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are added together, find the magnitude of the **resultant vector**.



Solution:

$$\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{EF}+\overrightarrow{GH}$$

$$=\begin{pmatrix}0\\3\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}-4\\-2\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$$

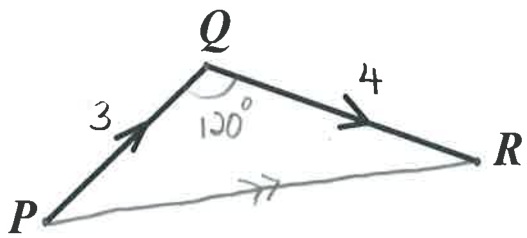
Magnitude of the resultant vector $=\sqrt{2^2+0^2}=2$.

Example 6

The diagram shows two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.

Given that $|\overrightarrow{PQ}|=3$, $|\overrightarrow{QR}|=4$ and angle PQR is 120°,

find the magnitude of the resultant vector $\overrightarrow{PR}$.



Solution:

Using Cosine rule,

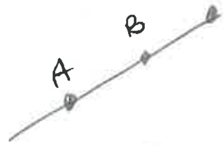
$$|\overrightarrow{PR}|^2=|\overrightarrow{PQ}|^2+|\overrightarrow{QR}|^2-2|\overrightarrow{PQ}||\overrightarrow{QR}|\cos 120^\circ$$

$$|\overrightarrow{PR}|=\sqrt{3^2+4^2-2(3)(4)\cos 120^\circ}=\sqrt{37}\approx 6.08 \text{ units}$$

Example 7 $\overrightarrow{OP}-\overrightarrow{OR}$

(a) Express $\overrightarrow{RP}-\overrightarrow{TS}+\overrightarrow{SR}-\overrightarrow{SP}+\overrightarrow{TQ}$ as a single vector.
$\overrightarrow{OS}-\overrightarrow{OT}$

(b) Three points A, B and C have position vectors given by $\underset{\overrightarrow{OA}}{\underbrace{\underset{\sim}{p}-4\underset{\sim}{q}}}$, $\underset{\overrightarrow{OB}}{\underbrace{2\underset{\sim}{p}-\underset{\sim}{q}}}$ and $\underset{\overrightarrow{OC}}{\underbrace{4\underset{\sim}{p}+5\underset{\sim}{q}}}$ respectively. C
Using vector method, show that the three points are collinear. $\overrightarrow{AB}=\lambda\overrightarrow{BC}$



Solution:

(a) Decompose the vectors into their position vectors:

$$\overrightarrow{RP}-\overrightarrow{TS}+\overrightarrow{SR}-\overrightarrow{SP}+\overrightarrow{TQ}=\left(\overrightarrow{OP}-\overrightarrow{OR}\right)-\left(\overrightarrow{OS}-\overrightarrow{OT}\right)+\left(\overrightarrow{OR}-\overrightarrow{OS}\right)-\left(\overrightarrow{OP}-\overrightarrow{OS}\right)+\left(\overrightarrow{OQ}-\overrightarrow{OT}\right)=\overrightarrow{OQ}-\overrightarrow{OS}=\overrightarrow{SQ}$$

(b)

$$\begin{pmatrix}2\\-1\end{pmatrix}-\begin{pmatrix}1\\-4\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(2\underset{\sim}{p}-\underset{\sim}{q}\right)-\left(\underset{\sim}{p}-4\underset{\sim}{q}\right)=\underset{\sim}{p}+3\underset{\sim}{q}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\left(4\underset{\sim}{p}+5\underset{\sim}{q}\right)-\left(2\underset{\sim}{p}-\underset{\sim}{q}\right)=2\underset{\sim}{p}+6\underset{\sim}{q}=2\left(\underset{\sim}{p}+3\underset{\sim}{q}\right)=2\overrightarrow{AB}$$

Since $\overrightarrow{BC}=2\overrightarrow{AB}$, the three points A, B and C are collinear.

Example 8: Given that $\overrightarrow{OA} = 2\underset{\sim}{r} + (\alpha - 1)\underset{\sim}{s}$, $\overrightarrow{OB} = 3\underset{\sim}{s} - 5\underset{\sim}{r}$ and $\overrightarrow{OC} = (2-\alpha)\underset{\sim}{r} - \underset{\sim}{s}$, where ~~the two vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are not parallel and~~ α is non-zero. Find the value of α if the three points are collinear.

Solution:

$\overrightarrow{AB} = -7\underset{\sim}{r} + (4-\alpha)\underset{\sim}{s}, \overrightarrow{AC} = -\alpha\underset{\sim}{r} - \alpha\underset{\sim}{s}$

$\overrightarrow{AB} = m\overrightarrow{AC} \Rightarrow \begin{pmatrix} -7 \\ 4-\alpha \end{pmatrix} = m\begin{pmatrix} -\alpha \\ -\alpha \end{pmatrix}$

$m\alpha = 7, \alpha = \dfrac{7}{m}$

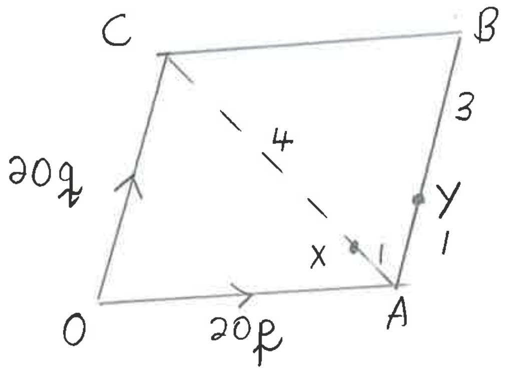
$4 - \alpha = -\dfrac{7}{\alpha} \cdot \alpha$

$\alpha = 11$

Example 9: $OABC$ is a parallelogram. The point X on AC is such that $AX = \frac{1}{5}AC$. The point Y on AB is such that $AY = \frac{1}{4}AB$. Given that $\overrightarrow{OA} = 20\underset{\sim}{p}$, and $\overrightarrow{OC} = 20\underset{\sim}{q}$, express in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$

(a) $\overrightarrow{AC}$, (b) $\overrightarrow{AX}$, (c) $\overrightarrow{OX}$, (d) $\overrightarrow{OY}$.

What do the results of (c) and (d) tell you about O, X and Y?



Solution:

$\overrightarrow{AC}, = 20\underset{\sim}{q} - 20\underset{\sim}{p}, \overrightarrow{AX} = 4\underset{\sim}{q} - 4\underset{\sim}{p},$

$\overrightarrow{OX}, = 20\underset{\sim}{p} + (4\underset{\sim}{q} - 4\underset{\sim}{p}) = 16\underset{\sim}{p} + 4\underset{\sim}{q}$

$\overrightarrow{OY} = 20\underset{\sim}{p} + \frac{1}{4}(20\underset{\sim}{q}) = 20\underset{\sim}{p} + 5\underset{\sim}{q}$

Since $\overrightarrow{OX} = m(\overrightarrow{OY})$, O, X and Y are collinear points

Example 10: Given that $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} n \\ 13 \end{pmatrix}$.

(a) Express $2\overrightarrow{AB} + 5\overrightarrow{BC}$ as a column vector and hence find its magnitude.

(b) Given that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, find the value of n.

Solution:

$2\overrightarrow{AB} + 5\overrightarrow{BC} = \begin{pmatrix} -4 \\ 18 \end{pmatrix} + \begin{pmatrix} 15 \\ 25 \end{pmatrix} = \begin{pmatrix} 11 \\ 43 \end{pmatrix}$

$\left|2\overrightarrow{AB} + 5\overrightarrow{BC}\right| = \sqrt{1970} \approx 44.4$

$m\begin{pmatrix} -2 \\ 9 \end{pmatrix} = \begin{pmatrix} n \\ 13 \end{pmatrix}$

$9m = 13$

$m = \dfrac{13}{9} \Rightarrow n = -2\dfrac{8}{9}$

Example 11

Given that $\overrightarrow{AB}=\begin{pmatrix}-2\\9\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}3\\5\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}n\\13\end{pmatrix}$.

(a) Express $2\overrightarrow{AB}+5\overrightarrow{BC}$ as a column vector and hence find its magnitude.

(b) Given that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, find the value of n.

Solution:

(a)

$$2\overrightarrow{AB}+5\overrightarrow{BC}=2\begin{pmatrix}-2\\9\end{pmatrix}+5\begin{pmatrix}3\\5\end{pmatrix}=\begin{pmatrix}-4\\18\end{pmatrix}+\begin{pmatrix}15\\25\end{pmatrix}=\begin{pmatrix}11\\43\end{pmatrix}$$

$$\left|2\overrightarrow{AB}+5\overrightarrow{BC}\right|=\sqrt{11^2+43^2}=\sqrt{1970}\approx 44.4$$

(b)

$$\overrightarrow{CD}=m\overrightarrow{AB}$$

$$\begin{pmatrix}n\\13\end{pmatrix}=m\begin{pmatrix}-2\\9\end{pmatrix}$$

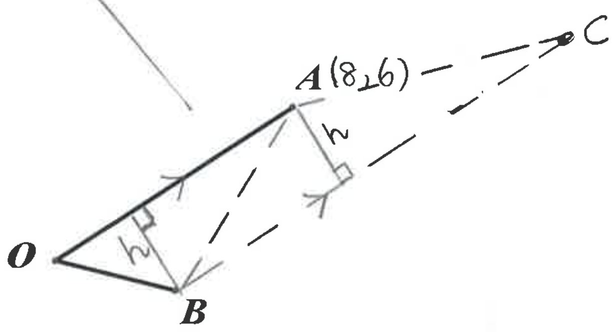
$$\left.\begin{matrix}n=-2m\\13=9m\end{matrix}\right\}n=-\frac{26}{9}$$

Example 12

The diagram shows three points O, A and B where $A\ (8, 6)$ and $\overrightarrow{BC}=\begin{pmatrix}12\\k\end{pmatrix}$. Given that BC is parallel to OA, find

(i) the value of k,

(ii) the ratio $\dfrac{\text{area of }\Delta OAB}{\text{area of }\Delta ACB}$.



Solution:

(i) BC is parallel to OA

$$\overrightarrow{BC}=m\overrightarrow{OA}$$

$$\begin{pmatrix}12\\k\end{pmatrix}=m\begin{pmatrix}8\\6\end{pmatrix},$$

$$m=\frac{3}{2},\ k=9$$

(ii) From (i), $\overrightarrow{BC}=\frac{3}{2}\overrightarrow{OA}\rightarrow\dfrac{|\overrightarrow{BC}|}{|\overrightarrow{OA}|}=\dfrac{3}{2}$

hence $\dfrac{\text{area of }\Delta OAB}{\text{area of }\Delta ACB}=\dfrac{|\overrightarrow{OA}|}{|\overrightarrow{BC}|}=\dfrac{2}{3}$ [Δ OAB and Δ ACB have common height]

$\ldots \underset{\sim}{a} + \ldots \underset{\sim}{b}$

$\ldots \underset{\sim}{a} + \ldots \underset{\sim}{b}$

Note:

(1) For triangles with common height, their **area ratio** is equal to their **base ratio**.

(2) For triangles with common base, their **area ratio** is equal to their **height ratio**.

(3) For triangles with no common height or base, their **area ratio** is equal to their **base × height ratio**.

(4) For similar triangles, their **area ratio** is equal to the **square** of their height ratio or base ratio or ratio of any corresponding sides.

Example 13

Given $\overrightarrow{PQ} = -5\underset{\sim}{i} + 6\underset{\sim}{j}$, $\overrightarrow{QR} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\overrightarrow{RS} = m\underset{\sim}{i} + 12\underset{\sim}{j}$.

(i) Express $2\overrightarrow{PQ} - 10\overrightarrow{QR}$ in terms of $\underset{\sim}{i}$ and/or $\underset{\sim}{j}$.

(ii) Given that $\overrightarrow{RS}$ is parallel to $\overrightarrow{PQ}$, find the value of m.

(iii) Find $|\overrightarrow{PQ}|$.

Solution:

(i)

$$2\overrightarrow{PQ} - 10\overrightarrow{QR} = 2\begin{pmatrix}-5\\6\end{pmatrix} - 10\begin{pmatrix}1\\-3\end{pmatrix}$$

$$= \begin{pmatrix}-10\\12\end{pmatrix} - \begin{pmatrix}10\\-30\end{pmatrix}$$

$$= \begin{pmatrix}-20\\42\end{pmatrix}$$

$$= -20\underset{\sim}{i} + \overset{42}{\cancel{4}}\underset{\sim}{j}$$

(ii)

$$\overrightarrow{RS} = n\overrightarrow{PQ}$$

$$\begin{pmatrix}m\\12\end{pmatrix} = n\begin{pmatrix}-5\\6\end{pmatrix}$$

$$\begin{pmatrix}m\\12\end{pmatrix} = \begin{pmatrix}-5n\\6n\end{pmatrix}$$

$$\left.\begin{aligned} m &= -5n \\ 12 &= 6n \end{aligned}\right\} m = -10$$

(iii) $|\overrightarrow{PQ}| = \sqrt{(-5)^2 + (6)^2} = \sqrt{61}$.

Example 14

Two points P and Q have position vectors given by $\overbrace{\underset{\sim}{i} + 8\underset{\sim}{j}}^{\overrightarrow{OP}}$ and $\overbrace{7\underset{\sim}{i} - 4\underset{\sim}{j}}^{\overrightarrow{OQ}}$ respectively. The point R divides the line PQ in the ratio 1 : 5. Find the coordinates of R.

Solution:

Let the coordinates of R be (a, b)

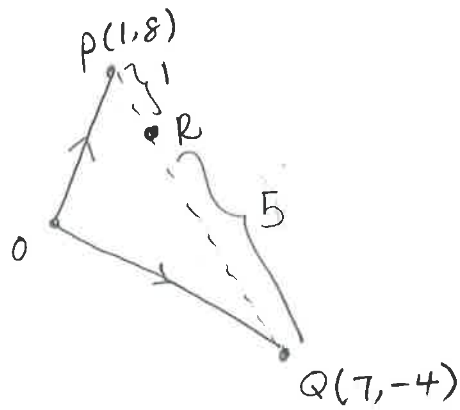
$$\overrightarrow{PQ} = 6\overrightarrow{PR}$$

$$\overrightarrow{OQ} - \overrightarrow{OP} = 6\left(\overrightarrow{OR} - \overrightarrow{OP}\right)$$

$$\begin{pmatrix}7\\-4\end{pmatrix} - \begin{pmatrix}1\\8\end{pmatrix} = 6\begin{pmatrix}a\\b\end{pmatrix} - 6\begin{pmatrix}1\\8\end{pmatrix}$$

$$\begin{pmatrix}6\\-12\end{pmatrix} = \begin{pmatrix}6a-6\\6b-48\end{pmatrix}$$

$$\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}2\\6\end{pmatrix}$$

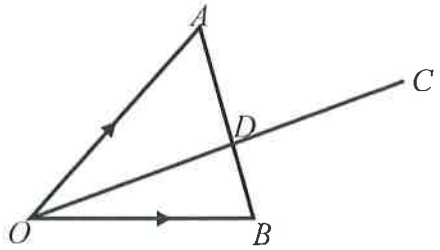


Coordinates of R is (2, 6)

Example 15: It is given that $\overrightarrow{OA} = 2\underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. The line OC cuts AB at D such that $3\overrightarrow{OC} = 4\underset{\sim}{a} + 2\underset{\sim}{b}$. If $\overrightarrow{AD} = \lambda\,\overrightarrow{AB}$ and $\overrightarrow{OD} = \mu\,\overrightarrow{OC}$, express $\overrightarrow{OD}$ in terms of

(a) λ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, (b) μ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Hence find the values of λ and μ.



Solution:

$$\overrightarrow{OD} = \frac{\lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}}{1} = \lambda\underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$$

$$\overrightarrow{OD} = \mu\cdot\frac{1}{3}(4\underset{\sim}{a} + 2\underset{\sim}{b}) = \frac{4}{3}\mu\underset{\sim}{a} + \frac{2}{3}\mu\underset{\sim}{b}$$

Comparing,

$$\lambda = \frac{4}{3}\mu \rightarrow (1)$$

$$(1-\lambda) = \frac{2}{3}\mu \rightarrow (2)$$

$$\mu = \frac{1}{2}, \lambda = \cancel{\frac{2}{3}}\ \frac{3}{4}$$

Example 16: In the diagram, the lines ST, PQ and OR are parallel. OPS, OQT and RQS are straight lines.

(a) Given that $\overrightarrow{OR} = 3\underset{\sim}{a}$, $\overrightarrow{OP} = \underset{\sim}{b}$, $\overrightarrow{PS} = 2\underset{\sim}{b}$ and $\overrightarrow{OQ} = 2\underset{\sim}{a} + \underset{\sim}{b}$ express, in terms of $\underset{\sim}{a}$ and/or $\underset{\sim}{b}$,

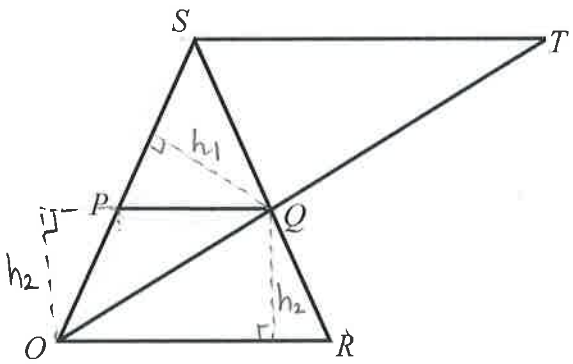
(i) $\overrightarrow{PQ}$, (ii) $\overrightarrow{QR}$, (iii) $\overrightarrow{QT}$, (iv) $\overrightarrow{ST}$

(b) Find the numerical value of

(i) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta SPQ}$ Common ht

(ii) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta OST}$ Similar Δs

(iii) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta ORQ}$ Common ht



Solution:

$\overrightarrow{PQ} = 2\underset{\sim}{a}$

$\overrightarrow{QR} = \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{QT} = 2(2\underset{\sim}{a} + \underset{\sim}{b}) = 4\underset{\sim}{a} + 2\underset{\sim}{b}$

$\overrightarrow{ST} = 3(2\underset{\sim}{a}) = 6\underset{\sim}{a}$

b(i) $\frac{1}{2}$ (ii) $\frac{1}{9}$ (iii) $\frac{2}{3}$

b(i) $\dfrac{\frac{1}{2} \times OP \times h_1}{\frac{1}{2} \times SP \times h_1} = \dfrac{OP}{SP}$

(ii) $\Delta OPQ \sim \Delta OST$

$\therefore \dfrac{A_1}{A_2} = \left(\dfrac{\ell_1}{\ell_2}\right)^2$

(iii) $\dfrac{\frac{1}{2} \times PQ \times h_2}{\frac{1}{2} \times OR \times h_2} = \dfrac{PQ}{OR}$

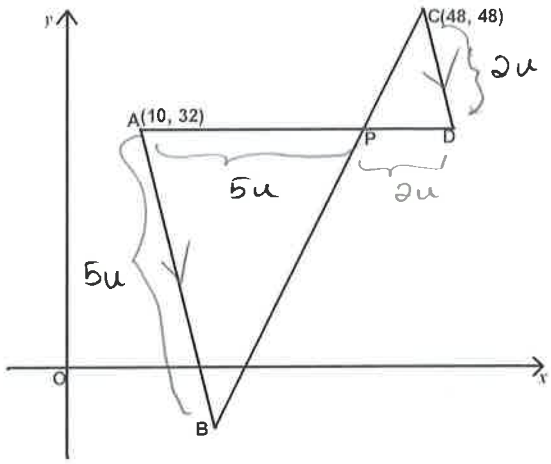
Example 17: In the diagram, A is the point (10, 32), C is the point (48, 48) and that $\overrightarrow{AB} = \frac{5}{2}\overrightarrow{CD}$ and $\overrightarrow{AB} = \begin{pmatrix} 15 \\ -40 \end{pmatrix}$.

(a) By means of a vector approach, find

(i) the coordinates of B, (ii) the coordinates of D, (iii) $\overrightarrow{BC}$.

(b) Given also APD and BPC are straight lines, find the coordinates of P.

(c) Find $\dfrac{\text{area of } \Delta PCD}{\text{area of } \Delta APB}$. Similar Δs



Solution:

(a)

$$\overrightarrow{OB} = \overrightarrow{AB} + \overrightarrow{OA}$$

$$= \begin{pmatrix} 25 \\ -8 \end{pmatrix} \quad \therefore B \text{ is } (25, -8)$$

$$\overrightarrow{OD} = \overrightarrow{CD} + \overrightarrow{OC} = \frac{2}{5}\begin{pmatrix} 15 \\ -40 \end{pmatrix} + \begin{pmatrix} 48 \\ 48 \end{pmatrix}$$

$$= \begin{pmatrix} 54 \\ 32 \end{pmatrix} \quad \therefore D \text{ is } (54, 32)$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB}$$

$$= \begin{pmatrix} 23 \\ 56 \end{pmatrix}$$

(b) $\dfrac{|\overrightarrow{AB}|}{|\overrightarrow{CD}|}$

$$\text{ratio} = \frac{\sqrt{15^2 + 40^2}}{\sqrt{6^2 + 16^2}} = \frac{5}{2} \text{ (given)}$$

$$P = \left(10 + \frac{5}{7}(54 - 10), 32\right)$$

$$= \left(41\frac{3}{7}, 32\right)$$

Alt: use eq^n of str. line
& find pt. of intersection.

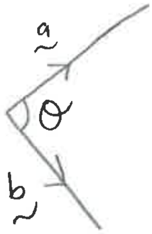
(c)

$$\text{Area} = \left(\frac{2}{5}\right)^2$$

$$= \frac{4}{25}$$

Example 18: Find the angle between the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

Solution:



$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\end{pmatrix}}{\left(\sqrt{3^2+1^2}\right)\left(\sqrt{2^2+(-3)^2}\right)}$$

$$= \frac{6-3}{\sqrt{10}\sqrt{13}}$$

$$= \frac{3}{\sqrt{130}}$$

$$\theta = 74.7^\circ$$

Example 19:

(a) If $\underset{\sim}{a} \perp \underset{\sim}{b}$, show that $|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$

(b) Given that $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, and $|\underset{\sim}{b}| = \sqrt{2}\,|\underset{\sim}{a}|$, show that $2\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

Solution:

(a)

$$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$\therefore |\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$$

(b)

$$(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a} = 0 \quad \text{since } \underset{\sim}{a}+\underset{\sim}{b}\perp\underset{\sim}{a}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{a} = 0$$

$$\underset{\sim}{b}\cdot\underset{\sim}{a} = -|\underset{\sim}{a}|^2$$

$$(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{b} = 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$$

$$= -2|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

$$= -2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}|^2 \quad \left[\text{Given } |\underset{\sim}{b}| = \sqrt{2}\,|\underset{\sim}{a}|\right]$$

$$= 0$$

$$\therefore 2\underset{\sim}{a}+\underset{\sim}{b} \perp \underset{\sim}{b}$$

Example 20: If $3\underset{\sim}{i} + \lambda\underset{\sim}{j}$ and $2\lambda^2\underset{\sim}{i} - \underset{\sim}{j}$ are perpendicular vectors, find the value(s) of λ.

$\underset{\sim}{a}\cdot\underset{\sim}{b} = 0$

Solution:

$$(3\underset{\sim}{i} + \lambda\underset{\sim}{j})\cdot(2\lambda^2\underset{\sim}{i} - \underset{\sim}{j}) = 0$$

$$\begin{pmatrix}3\\\lambda\end{pmatrix}\cdot\begin{pmatrix}2\lambda^2\\-1\end{pmatrix} = 0$$

$$6\lambda^2 - \lambda = 0$$

$$\lambda = 0 \text{ or } \frac{1}{6}$$