Topic: Vectors

Scalar Versus Vector

All physical quantities can be classified into scalar or vector.

A ***scalar*** quantity possesses only ***magnitude***.
e.g. mass, distance, speed, work, energy time, temperature, length, area, volume.

A ***vector*** quantity possesses both ***magnitude*** and ***direction***.
e.g. force, displacement, velocity, acceleration, momentum, moment.

Representation of Vectors

Geometrically, a vector can be represented by a ***directed line segment***. *The arrow indicates its direction while the length is proportional to the vector's magnitude.*

The diagram shows a point O being displaced to a point A, the displacement can be represent by several notations such as $\overrightarrow{OA}$, $\mathbf{a}$ (printed material-use-only) or $\underset{\sim}{a}$.

Equal Vectors, Negative Vectors, Zero (or null) Vectors

(i) If two vectors have the ***same magnitude*** and the ***same direction***, they are ***equal*** and can be represented by the same notation $\underset{\sim}{a}$.

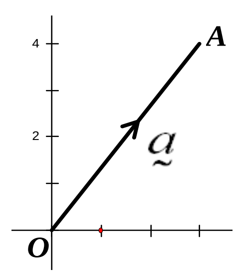
(ii) ***Negative vectors*** have the same magnitude but opposite directions.
e.g. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are ***negative*** vectors of each other and we write $\overrightarrow{AB} = -\overrightarrow{BA}$.

(iii) $\overrightarrow{AB} - \overrightarrow{AB} = \underset{\sim}{0}$, $\underset{\sim}{0}$ is a ***zero vector*** . It has zero magnitude and its direction is indeterminate.

Position Vectors

A position vector defines the position of one point relative to another point.

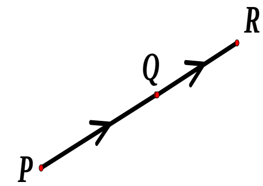
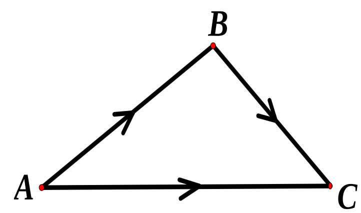
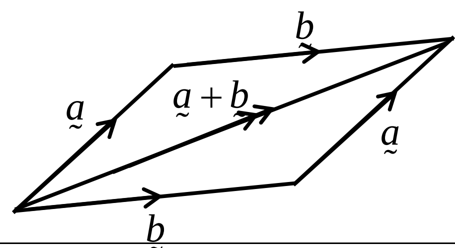
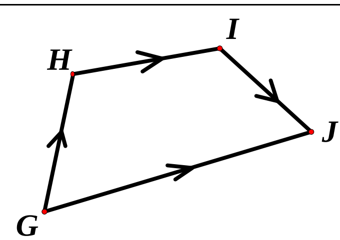
If O is the origin and the coordinates of A is (3, 4), then $\overrightarrow{OA}$ can be written as a ***position (or column) vector***, $\underset{\sim}{a} = \overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.



If X has coordinates (m, n) then $\overrightarrow{OX} = \begin{pmatrix} m \\ n \end{pmatrix}$, conversely if $\overrightarrow{OY} = \begin{pmatrix} p \\ q \end{pmatrix}$ then the coordinates of Y is (p, q).

Law of Addition of Coplanar Vectors

Coplanar vectors are vectors lying on the same plane.

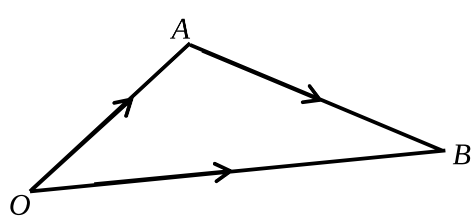
Case 1 : Addition of two vectors that are **parallel**: $\overrightarrow{PQ}+\overrightarrow{QR}=\overrightarrow{PR}$ Adding vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$ give rise to **resultant vector** $\overrightarrow{PR}$. We can write $\lvert\overrightarrow{PQ}\rvert+\lvert\overrightarrow{QR}\rvert=\lvert\overrightarrow{PR}\rvert$.	
Case 2 : Addition of two vectors that are **non-parallel:** $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$ Note that here $\lvert\overrightarrow{AB}\rvert+\lvert\overrightarrow{BC}\rvert\neq\lvert\overrightarrow{AC}\rvert$. (Head-to-tail method / Triangular law)	
Case 3 : Addition of more than two non-parallel vectors $\overrightarrow{GH}+\overrightarrow{HI}+\overrightarrow{IJ}=\overrightarrow{GJ}$ Again note that here $\lvert\overrightarrow{GH}\rvert+\lvert\overrightarrow{HI}\rvert+\lvert\overrightarrow{IJ}\rvert\neq\lvert\overrightarrow{GJ}\rvert$. This is an extension of Case 2. In a parallelogram, the diagonal $\underset{\sim}{a}+\underset{\sim}{b}$ can be found using the law of addition: 	

Note: Magnitudes of non-parallel vectors cannot be added and subtracted by the usual rules of arithmetic.

All vectors can be expressed in terms of position vectors eg : $\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}$, $\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}$

Using the law of vector addition,

$$\begin{aligned}\overrightarrow{AB}&=\overrightarrow{AO}+\overrightarrow{OB}\\&=-\overrightarrow{OA}+\overrightarrow{OB}\\&=\overrightarrow{OB}-\overrightarrow{OA}\end{aligned}$$



Magnitude of a vector

The ***magnitude*** of $\overrightarrow{OA}$ is denoted by $|\overrightarrow{OA}|$ or $|\underset{\sim}{a}|$. By Pythagoras Theorem, $|\overrightarrow{OA}| = \sqrt{3^2 + 4^2} = 5$.

For any vector with column vector $\begin{pmatrix} p \\ q \end{pmatrix}$, its magnitude is given by $\sqrt{p^2 + q^2}$.

Unit vector

Vector whose magnitude is 1. Unit vectors are used to specify directions. The unit vector in the direction of $\underset{\sim}{a}$ vector is denoted by $\hat{\underset{\sim}{a}} = \dfrac{\underset{\sim}{a}}{|\underset{\sim}{a}|}$.

The i, j notations

$\underset{\sim}{i}$, $\underset{\sim}{j}$ are unit vectors in the direction of the positive x – and y – axis respectively,

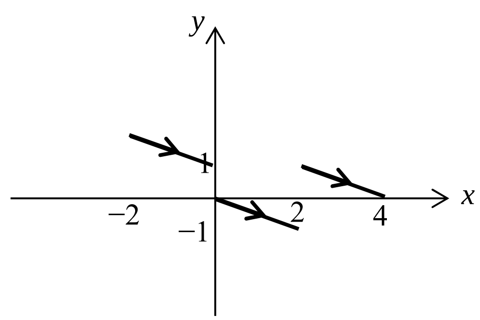
i.e. $\underset{\sim}{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\underset{\sim}{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

They are known as unit vector as their magnitude is 1, that is $|\underset{\sim}{i}| = |\underset{\sim}{j}| = 1$.

All vectors can be expressed in terms of $\underset{\sim}{i}$, $\underset{\sim}{j}$,

Recall column vector $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, then $\overrightarrow{PQ} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 2\underset{\sim}{i} + \underset{\sim}{j}$.

All vectors which have the same magnitude and direction as $\overrightarrow{PQ}$ can be represented by any of the directed line segments:



Generally $\overrightarrow{AB} = \begin{pmatrix} a \\ b \end{pmatrix} = a\underset{\sim}{i} + b\underset{\sim}{j}$.

Product of a Scalar and a Vector

For scalar $\lambda > 0$, then $\lambda\underset{\sim}{a}$ is a vector with magnitude λ times that of $\underset{\sim}{a}$ and in the ***same*** direction.
For scalar $\lambda < 0$, then $\lambda\underset{\sim}{a}$ is a vector with magnitude λ times that of $\underset{\sim}{a}$ and is in ***opposite*** direction.

eg. If $\overrightarrow{AB} = -8\underset{\sim}{p} + 6\underset{\sim}{q}$ and *CD* is parallel to *AB* and is half the size of *AB*, we can express vector $\overrightarrow{CD}$ as a ***scalar multiple*** of vector $\overrightarrow{AB}$**,** $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{AB} = -4\underset{\sim}{p} + 3\underset{\sim}{q}$**.**

Properties of Scalar Multiples:

(i) $\lambda(\mu \underset{\sim}{a}) = (\lambda\mu)\underset{\sim}{a}$

(ii) $\lambda(\underset{\sim}{a}+\underset{\sim}{b}) = \lambda\underset{\sim}{a}+\lambda\underset{\sim}{b}$

(iii) $(\lambda+\mu)\underset{\sim}{a} = \lambda\underset{\sim}{a}+\mu\underset{\sim}{a}$

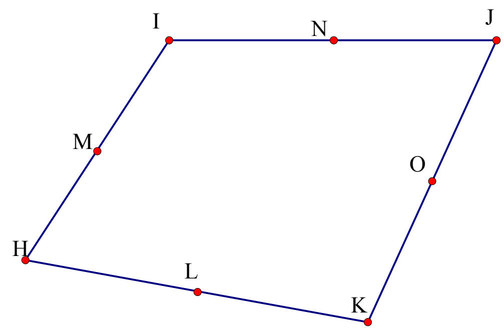
Examples of scalar multiples of vectors are

$displacement = time \times velocity$ where displacement and velocity are vectors, time is a scalar.

$force = mass \times acceleration$ where force and acceleration are vectors, mass is a scalar.

Midpoint Theorem

Given a quadrilateral *HIJK*, let *L*, *M*, *N*, and *O* denote the respective midpoints. The midpoints are joined to form a new shape *LMNO*.



Applying Midpoint Thm to triangle *HIJ*, $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{HJ}$

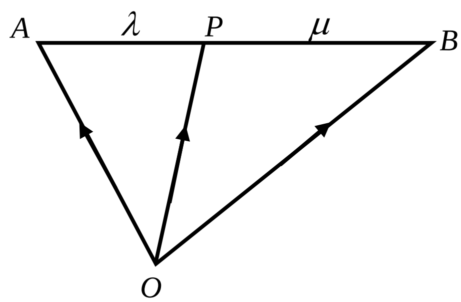
Applying Midpoint Thm to triangle *HKJ*, $\overrightarrow{LO} = \frac{1}{2}\overrightarrow{HJ}$

$\Rightarrow \overrightarrow{LO} = \overrightarrow{MN}$ and *LMNO* is a parallelogram

Ratio Theorem

If *P* lies between *A* and *B* such that $AP : PB = \lambda : \mu$, then

$$\overrightarrow{OP} = \frac{\mu \underset{\sim}{a}}{\lambda+\mu} + \frac{\lambda \underset{\sim}{b}}{\lambda+\mu} = \frac{1}{\lambda+\mu}(\mu\underset{\sim}{a}+\lambda\underset{\sim}{b})$$



Proof:

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\overrightarrow{AB}$$

$$= \overrightarrow{OA} + \frac{\lambda}{\lambda+\mu}\left(\overrightarrow{OB} - \overrightarrow{OA}\right)$$

$$= \frac{(\lambda+\mu)\overrightarrow{OA} + \lambda\overrightarrow{OB} - \lambda\overrightarrow{OA}}{\lambda+\mu}$$

$$= \frac{\lambda\overrightarrow{OA} + \mu\overrightarrow{OA} + \lambda\overrightarrow{OB} - \lambda\overrightarrow{OA}}{\lambda+\mu}$$

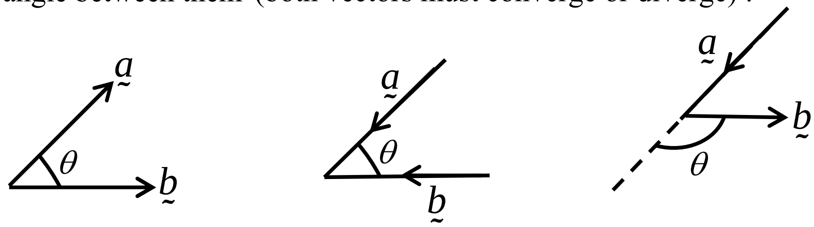
$$= \frac{\mu\overrightarrow{OA} + \lambda\overrightarrow{OB}}{\lambda+\mu}$$

If *P* is the mid-point of *AB*, then $AP : PB = 1:1$

By Ratio Theorem, $\overrightarrow{OP} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$

***Scalar / Dot Product**

The scaler product of two vectors, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, denoted by $\underset{\sim}{a}\cdot\underset{\sim}{b}$ is defined as $|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$ where θ is the angle between them (both vectors must converge or diverge) :



$$\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta}$$

Note:

(1) $\underset{\sim}{a}\cdot\underset{\sim}{b}$ is a scalar quantity

(2) $\underset{\sim}{a}\cdot\underset{\sim}{b}\cdot\underset{\sim}{c}$ has no meaning

(3) $(\underset{\sim}{a}\cdot\underset{\sim}{b})\underset{\sim}{c}$ is a vector parallel to $\underset{\sim}{c}$

(4) $\underset{\sim}{a}\cdot(\lambda\underset{\sim}{b})=\lambda(\underset{\sim}{a}\cdot\underset{\sim}{b}),\lambda\in\mathbb{R}$

(i) Parallel vectors

If $\underset{\sim}{a}\,/\!/\,\underset{\sim}{b}$ and $\underset{\sim}{a},\underset{\sim}{b}$ are in the same direction, then $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos 0=|\underset{\sim}{a}||\underset{\sim}{b}|$

If $\underset{\sim}{a}\,/\!/\,\underset{\sim}{b}$ and $\underset{\sim}{a},\underset{\sim}{b}$ are in the opposite direction, then $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\pi=-|\underset{\sim}{a}||\underset{\sim}{b}|$

Note that $\underset{\sim}{a}\cdot\underset{\sim}{a}=|\underset{\sim}{a}|^2\cos 0=|\underset{\sim}{a}|^2$

(ii) Perpendicular vectors

If $\underset{\sim}{a}\perp\underset{\sim}{b}$, then $\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\frac{\pi}{2}=0$

Conversely, $\underset{\sim}{a}\cdot\underset{\sim}{b}=0$, then either $\underset{\sim}{a}=0$, $\underset{\sim}{b}=0$ or $a\perp b$

(iii) Scaler product is commutative

$\underset{\sim}{a}\cdot\underset{\sim}{b}=|\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta=|\underset{\sim}{b}||\underset{\sim}{a}|\cos\theta=\underset{\sim}{b}\cdot\underset{\sim}{a}$

(iv) Scaler product is distributive over vector addition

$\underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c})=\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{a}\cdot\underset{\sim}{c}$

$(\underset{\sim}{b}+\underset{\sim}{c})\cdot\underset{\sim}{a}=\underset{\sim}{b}\cdot\underset{\sim}{a}+\underset{\sim}{c}\cdot\underset{\sim}{a}$

(v) $\underset{\sim}{a}=x_1\underset{\sim}{i}+y_1\underset{\sim}{j}$ and $\underset{\sim}{b}=x_2\underset{\sim}{i}+y_2\underset{\sim}{j}$ then $\boxed{\underset{\sim}{a}\cdot\underset{\sim}{b}=\begin{pmatrix}x_1\\y_1\end{pmatrix}\cdot\begin{pmatrix}x_2\\y_2\end{pmatrix}=x_1x_2+y_1y_2}$

(vi) Modulus and Scalar Product

From (i), $|\underset{\sim}{a}|^2=\underset{\sim}{a}\cdot\underset{\sim}{a}$

If $\underset{\sim}{c}=\underset{\sim}{a}-\underset{\sim}{b}$,

then $|\underset{\sim}{c}|^2=\underset{\sim}{c}\cdot\underset{\sim}{c}$

$=(\underset{\sim}{a}-\underset{\sim}{b})\cdot(\underset{\sim}{a}-\underset{\sim}{b})$

$=\underset{\sim}{a}\cdot\underset{\sim}{a}-\underset{\sim}{a}\cdot\underset{\sim}{b}-\underset{\sim}{b}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{b}$ (Distributive Law)

$=\underset{\sim}{a}\cdot\underset{\sim}{a}-2\underset{\sim}{a}\cdot\underset{\sim}{b}+\underset{\sim}{b}\cdot\underset{\sim}{b}$ (Commutative Law: $\underset{\sim}{a}\cdot\underset{\sim}{b}=\underset{\sim}{b}\cdot\underset{\sim}{a}$)

$=|\underset{\sim}{a}|^2-2\underset{\sim}{a}\cdot\underset{\sim}{b}+|\underset{\sim}{b}|^2$

$$\boxed{\therefore|\underset{\sim}{a}-\underset{\sim}{b}|^2=|\underset{\sim}{a}|^2-2\underset{\sim}{a}\cdot\underset{\sim}{b}+|\underset{\sim}{b}|^2}$$

Similarly, $\boxed{|\underset{\sim}{a}+\underset{\sim}{b}|^2=|\underset{\sim}{a}|^2+2\underset{\sim}{a}\cdot\underset{\sim}{b}+|\underset{\sim}{b}|^2}$

Example 1

State whether the following statement is true or false.

(i) $|\overrightarrow{AB}| = |\overrightarrow{CD}| \Rightarrow \overrightarrow{AB} = \overrightarrow{CD}$, (ii) $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow |\overrightarrow{AB}| = |\overrightarrow{CD}|$.

Solution: (i) False, though AB has the same magnitude as CD, they may not have the same direction.

(ii) True, if two vectors are equal, then both their magnitude and their direction must be the same.

Example 2

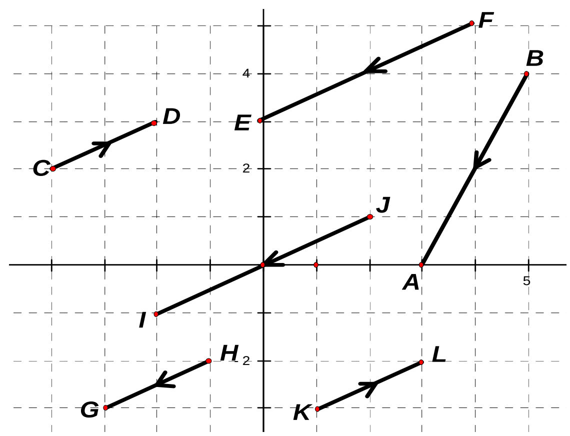
Given that $ABCD$ is a square,

(i) is $\overrightarrow{AB} = \overrightarrow{BC}$? (ii) is $\overrightarrow{AD} = \overrightarrow{BC}$? (iii) is $|\overrightarrow{AB}| = |\overrightarrow{BC}|$?

Solution: (i) No, different directions (ii) Yes (iii) Yes

Example 3

State whether the following statement is true or false.



(i) $\overrightarrow{AB} = \overrightarrow{EF}$,

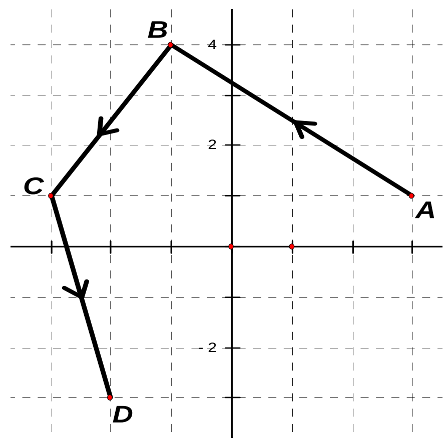
(ii) $|\overrightarrow{AB}| = |\overrightarrow{EF}| = |\overrightarrow{IJ}|$,

(iii) $\overrightarrow{CD} = \overrightarrow{KL} = -\overrightarrow{HG}$.

Solution: (i) False (ii) True (iii) True

Example 4

A boy kicked the ball from A to B, B to C and then from C to D.



(a) Express each of the following as a column vector.

(i) $\overrightarrow{AB}$, (ii) $\overrightarrow{BC}$, (iii) $\overrightarrow{CD}$, (iv) $\overrightarrow{AD}$.

(b) State whether the following statement is true or false.

(i) The sum of the magnitude $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}|$ represents the total distance traveled by the ball.

(ii) The vector $\overrightarrow{AD}$ represents the displacement of the ball.

(iii) $|\overrightarrow{AD}| = |\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}|$?

(iv) $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$?

Solution:

(a) (i) $\overrightarrow{AB} = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$, (ii) $\overrightarrow{BC} = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$, (iii) $\overrightarrow{CD} = \begin{pmatrix} 1 \\ -4 \end{pmatrix}$, (iv) $\overrightarrow{AD} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

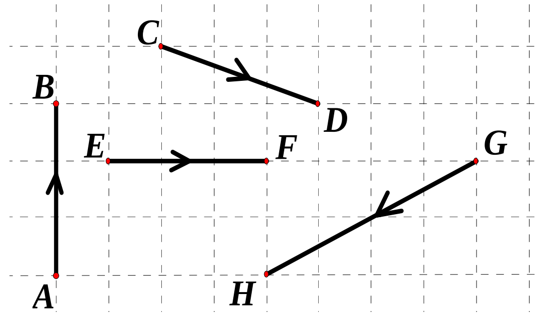
(b) (i) True, (ii) True,

(iii) False, because $|\overrightarrow{AD}| = \sqrt{5^2 + 4^2} = \sqrt{41}$, $|\overrightarrow{AB}| + |\overrightarrow{BC}| + |\overrightarrow{CD}| = 5 + \sqrt{13} + \sqrt{17}$

(iv) True $\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD} = \begin{pmatrix} -5 \\ -4 \end{pmatrix}$

Example 5

If vectors $\overrightarrow{AB}$, $\overrightarrow{CD}$, $\overrightarrow{EF}$ and $\overrightarrow{GH}$ are added together, find the magnitude of the **resultant vector**.



Solution:

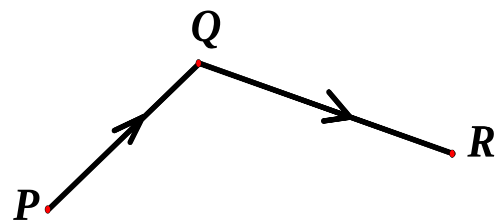
$$\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{EF}+\overrightarrow{GH}$$

$$=\begin{pmatrix}0\\3\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}-4\\-2\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$$

Magnitude of the resultant vector $=\sqrt{2^2+0^2}=2$.

Example 6

The diagram shows two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.



Given that $|\overrightarrow{PQ}|=3$, $|\overrightarrow{QR}|=4$ and angle PQR is 120º,

find the magnitude of the resultant vector $\overrightarrow{PR}$.

Solution:

Using Cosine rule,

$$|\overrightarrow{PR}|^2=|\overrightarrow{PQ}|^2+|\overrightarrow{QR}|^2-2|\overrightarrow{PQ}||\overrightarrow{QR}|\cos 120^\circ$$

$$|\overrightarrow{PR}|=\sqrt{3^2+4^2-2(3)(4)\cos 120^\circ}=\sqrt{37}\approx 6.08$$

Example 7

(a) Express $\overrightarrow{RP}-\overrightarrow{TS}+\overrightarrow{SR}-\overrightarrow{SP}+\overrightarrow{TQ}$ as a single vector.

(b) Three points A, B and C have position vectors given by $\underset{\sim}{p}-4\underset{\sim}{q}$, $2\underset{\sim}{p}-\underset{\sim}{q}$ and $4\underset{\sim}{p}+5\underset{\sim}{q}$ respectively. Using vector method, show that the three points are collinear.

Solution:

(a) Decompose the vectors into their position vectors:

$$\overrightarrow{RP}-\overrightarrow{TS}+\overrightarrow{SR}-\overrightarrow{SP}+\overrightarrow{TQ}=\left(\overrightarrow{OP}-\overrightarrow{OR}\right)-\left(\overrightarrow{OS}-\overrightarrow{OT}\right)+\left(\overrightarrow{OR}-\overrightarrow{OS}\right)-\left(\overrightarrow{OP}-\overrightarrow{OS}\right)+\left(\overrightarrow{OQ}-\overrightarrow{OT}\right)=\overrightarrow{OQ}-\overrightarrow{OS}=\overrightarrow{SQ}$$

(b)

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\left(2\underset{\sim}{p}-\underset{\sim}{q}\right)-\left(\underset{\sim}{p}-4\underset{\sim}{q}\right)=\underset{\sim}{p}+3\underset{\sim}{q}$$

$$\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=\left(4\underset{\sim}{p}+5\underset{\sim}{q}\right)-\left(2\underset{\sim}{p}-\underset{\sim}{q}\right)=2\underset{\sim}{p}+6\underset{\sim}{q}=2\left(\underset{\sim}{p}+3\underset{\sim}{q}\right)=2\overrightarrow{AB}$$

Since $\overrightarrow{BC}=2\overrightarrow{AB}$, the three points A, B and C are collinear.

Example 8: Given that $\overrightarrow{OA} = 2\underset{\sim}{r} + (\alpha - 1)\underset{\sim}{s}$, $\overrightarrow{OB} = 3\underset{\sim}{s} - 5\underset{\sim}{r}$ and $\overrightarrow{OC} = (2-\alpha)\underset{\sim}{r} - \underset{\sim}{s}$, where the two vectors $\underset{\sim}{r}$ and $\underset{\sim}{s}$ are not parallel and non-zero. Find the value of α if the three points are collinear.

Solution:

$\overrightarrow{AB} = -7\underset{\sim}{r} + (4-\alpha)\underset{\sim}{s}, \overrightarrow{AC} = -\alpha\underset{\sim}{r} - \alpha\underset{\sim}{s}$

$\overrightarrow{AB} = m\,\overrightarrow{AC}$

$m\alpha = 7, \alpha = \frac{7}{m}$

$4 - \alpha = -\frac{7}{\alpha}\cdot\alpha$

$\alpha = 11$

Example 9: *OABC* is a parallelogram. The point *X* on *AC* is such that $AX = \frac{1}{5}AC$. The point *Y* on *AB* is such that $AY = \frac{1}{4}AB$. Given that $\overrightarrow{OA} = 20\underset{\sim}{p}$, and $\overrightarrow{OC} = 20\underset{\sim}{q}$, express in terms of $\underset{\sim}{p}$ and $\underset{\sim}{q}$

(a) $\overrightarrow{AC}$, (b) $\overrightarrow{AX}$, (c) $\overrightarrow{OX}$, (d) $\overrightarrow{OY}$.

What do the results of (c) and (d) tell you about *O*, *X* and *Y*?

Solution:

$\overrightarrow{AC}, = 20\underset{\sim}{q} - 20\underset{\sim}{p}, \overrightarrow{AX} = 4\underset{\sim}{q} - 4\underset{\sim}{p},$

$\overrightarrow{OX}, = 20\underset{\sim}{p} + (4\underset{\sim}{q} - 4\underset{\sim}{p}) = 16\underset{\sim}{p} + 4\underset{\sim}{q}$

$\overrightarrow{OY} = 20\underset{\sim}{p} + \frac{1}{4}(20\underset{\sim}{q}) = 20\underset{\sim}{p} + 5\underset{\sim}{q}$

Since $\overrightarrow{OX} = m(\overrightarrow{OY})$, O, X and Y are collinear points

Example 10: Given that $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} n \\ 13 \end{pmatrix}$.

(a) Express $2\overrightarrow{AB} + 5\overrightarrow{BC}$ as a column vector and hence find its magnitude.

(b) Given that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, find the value of n.

Solution:

$2\overrightarrow{AB} + 5\overrightarrow{BC} = \begin{pmatrix} -4 \\ 18 \end{pmatrix} + \begin{pmatrix} 15 \\ 25 \end{pmatrix} = \begin{pmatrix} 11 \\ 43 \end{pmatrix}$

$\left|2\overrightarrow{AB} + 5\overrightarrow{BC}\right| = \sqrt{1970} \approx 44.4$

$m\begin{pmatrix} -2 \\ 9 \end{pmatrix} = \begin{pmatrix} n \\ 13 \end{pmatrix}$

$9m = 13$

$m = \frac{13}{9} \Rightarrow n = -2\frac{8}{9}$

Example 11

Given that $\overrightarrow{AB}=\begin{pmatrix}-2\\9\end{pmatrix}$, $\overrightarrow{BC}=\begin{pmatrix}3\\5\end{pmatrix}$, $\overrightarrow{CD}=\begin{pmatrix}n\\13\end{pmatrix}$.

(a) Express $2\overrightarrow{AB}+5\overrightarrow{BC}$ as a column vector and hence find its magnitude.

(b) Given that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, find the value of *n*.

Solution:
(a)

$$2\overrightarrow{AB}+5\overrightarrow{BC}=2\begin{pmatrix}-2\\9\end{pmatrix}+5\begin{pmatrix}3\\5\end{pmatrix}$$
$$=\begin{pmatrix}-4\\18\end{pmatrix}+\begin{pmatrix}15\\25\end{pmatrix}$$
$$=\begin{pmatrix}11\\43\end{pmatrix}$$

$$\left|2\overrightarrow{AB}+5\overrightarrow{BC}\right|=\sqrt{11^2+43^2}=\sqrt{1970}\approx 44.4$$

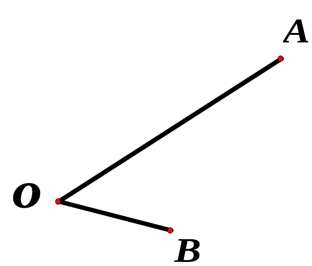
(b)

$$\overrightarrow{CD}=m\overrightarrow{AB}$$
$$\begin{pmatrix}n\\13\end{pmatrix}=m\begin{pmatrix}-2\\9\end{pmatrix}$$

$$\left.\begin{matrix}n=-2m\\13=9m\end{matrix}\right\}n=-\frac{26}{9}$$

Example 12

The diagram shows three points *O*, *A* and *B* where *A* (8, 6) and $\overrightarrow{BC}=\begin{pmatrix}12\\k\end{pmatrix}$. Given that *BC* is parallel to *OA*, find



(i) the value of *k*,

(ii) the ratio $\dfrac{\text{area of }\Delta OAB}{\text{area of }\Delta ACB}$.

Solution:
(i) *BC* is parallel to *OA*

$$\overrightarrow{BC}=m\overrightarrow{OA}$$
$$\begin{pmatrix}12\\k\end{pmatrix}=m\begin{pmatrix}8\\6\end{pmatrix},$$
$$m=\frac{3}{2},\ k=9$$

(ii) From (i), $\overrightarrow{BC}=\frac{3}{2}\overrightarrow{OA}\rightarrow\dfrac{|\overrightarrow{BC}|}{|\overrightarrow{OA}|}=\dfrac{3}{2}$

hence $\dfrac{\text{area of }\Delta OAB}{\text{area of }\Delta ACB}=\dfrac{|\overrightarrow{OA}|}{|\overrightarrow{BC}|}=\dfrac{2}{3}$ [Δ OAB and Δ ACB have common height]

Note:

(1) For triangles with common height, their **area ratio** is equal to their **base ratio**.

(2) For triangles with common base, their **area ratio** is equal to their **height ratio**.

(3) For triangles with no common height or base, their **area ratio** is equal to their **base × height ratio**.

(4) For similar triangles, their **area ratio** is equal to the **square** of their height ratio or base ratio or ratio of any corresponding sides.

Example 13

Given $\overrightarrow{PQ} = -5\underset{\sim}{i} + 6\underset{\sim}{j}$, $\overrightarrow{QR} = \underset{\sim}{i} - 3\underset{\sim}{j}$ and $\overrightarrow{RS} = m\underset{\sim}{i} + 12\underset{\sim}{j}$.

(i) Express $2\overrightarrow{PQ} - 10\overrightarrow{QR}$ in terms of $\underset{\sim}{i}$ and/or $\underset{\sim}{j}$.

(ii) Given that $\overrightarrow{RS}$ is parallel to $\overrightarrow{PQ}$, find the value of *m*.

(iii) Find $\left|\overrightarrow{PQ}\right|$.

Solution:

(i)

$$2\overrightarrow{PQ} - 10\overrightarrow{QR} = 2\begin{pmatrix} -5 \\ 6 \end{pmatrix} - 10\begin{pmatrix} 1 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} -10 \\ 12 \end{pmatrix} - \begin{pmatrix} 10 \\ -30 \end{pmatrix}$$
$$= \begin{pmatrix} -20 \\ 42 \end{pmatrix}$$
$$= -20\underset{\sim}{i} + 4\underset{\sim}{j}$$

(ii)

$$\overrightarrow{RS} = n\overrightarrow{PQ}$$
$$\begin{pmatrix} m \\ 12 \end{pmatrix} = n\begin{pmatrix} -5 \\ 6 \end{pmatrix}$$
$$\begin{pmatrix} m \\ 12 \end{pmatrix} = \begin{pmatrix} -5n \\ 6n \end{pmatrix}$$
$$\left.\begin{matrix} m = -5n \\ 12 = 6n \end{matrix}\right\} m = -10$$

(iii) $\left|\overrightarrow{PQ}\right| = \sqrt{(-5)^2 + (6)^2} = \sqrt{61}$.

Example 14

Two points *P* and *Q* have position vectors given by $\underset{\sim}{i} + 8\underset{\sim}{j}$ and $7\underset{\sim}{i} - 4\underset{\sim}{j}$ respectively. The point *R* divides the line *PQ* in the ratio 1 : 5. Find the coordinates of *R*.

Solution:

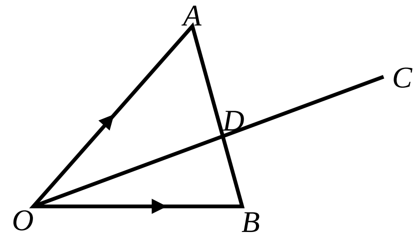
Let the coordinates of *R* be (*a*, *b*)

$$\overrightarrow{PQ} = 6\overrightarrow{PR}$$
$$\overrightarrow{OQ} - \overrightarrow{OP} = 6\left(\overrightarrow{OR} - \overrightarrow{OP}\right)$$

$$\begin{pmatrix} 7 \\ -4 \end{pmatrix} - \begin{pmatrix} 1 \\ 8 \end{pmatrix} = 6\begin{pmatrix} a \\ b \end{pmatrix} - 6\begin{pmatrix} 1 \\ 8 \end{pmatrix}$$
$$\begin{pmatrix} 6 \\ -12 \end{pmatrix} = \begin{pmatrix} 6a - 6 \\ 6b - 48 \end{pmatrix}$$
$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \end{pmatrix}$$

Coordinates of *R* is (2, 6)

Example 15: It is given that $\overrightarrow{OA} = 2\underset{\sim}{a}$ and $\overrightarrow{OB} = \underset{\sim}{b}$. The line OC cuts AB at D such that $3\overrightarrow{OC} = 4\underset{\sim}{a} + 2\underset{\sim}{b}$. If $\overrightarrow{AD} = \lambda \overrightarrow{AB}$ and $\overrightarrow{OD} = \mu \overrightarrow{OC}$, express $\overrightarrow{OD}$ in terms of



(a) λ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$, (b) μ, $\underset{\sim}{a}$ and $\underset{\sim}{b}$.

Hence find the values of λ and μ.

Solution:

$$\overrightarrow{OD} = \frac{\lambda \underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}}{1} = \lambda \underset{\sim}{a} + (1-\lambda)\underset{\sim}{b}$$

$$\overrightarrow{OD} = \mu \cdot \frac{1}{3}(4\underset{\sim}{a} + 2\underset{\sim}{b}) = \frac{4}{3}\mu \underset{\sim}{a} + \frac{2}{3}\mu \underset{\sim}{b}$$

Comparing,

$$\lambda = \frac{4}{3}\mu \rightarrow (1)$$

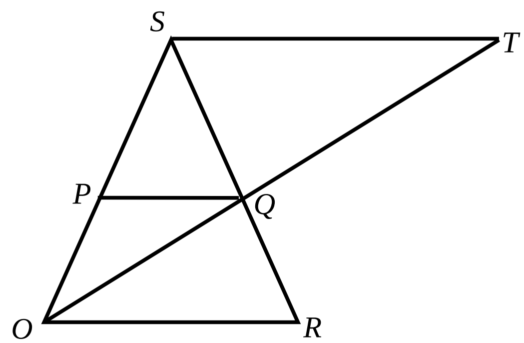
$$(1-\lambda) = \frac{2}{3}\mu \rightarrow (2)$$

$$\mu = \frac{1}{2}, \lambda = \frac{2}{3}$$

Example 16: In the diagram, the lines ST, PQ and OR are parallel. OPS, OQT and RQS are straight lines.

(a) Given that $\overrightarrow{OR} = 3\underset{\sim}{a}$, $\overrightarrow{OP} = \underset{\sim}{b}$, $\overrightarrow{PS} = 2\underset{\sim}{b}$ and $\overrightarrow{OQ} = 2\underset{\sim}{a} + \underset{\sim}{b}$ express, in terms of $\underset{\sim}{a}$ and/or $\underset{\sim}{b}$,

(i) $\overrightarrow{PQ}$, (ii) $\overrightarrow{QR}$, (iii) $\overrightarrow{QT}$, (iv) $\overrightarrow{ST}$



(b) Find the numerical value of

(i) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta SPQ}$

(ii) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta OST}$

(iii) $\dfrac{\text{Area of } \Delta OPQ}{\text{Area of } \Delta ORQ}$

Solution:

$\overrightarrow{PQ} = 2\underset{\sim}{a}$

$\overrightarrow{QR} = \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{QT} = 2(2\underset{\sim}{a} + \underset{\sim}{b}) = 4\underset{\sim}{a} + 2\underset{\sim}{b}$

$\overrightarrow{ST} = 3(2\underset{\sim}{a}) = 6\underset{\sim}{a}$

b(i) $\frac{1}{2}$ (ii) $\frac{1}{9}$ (iii) $\frac{2}{3}$

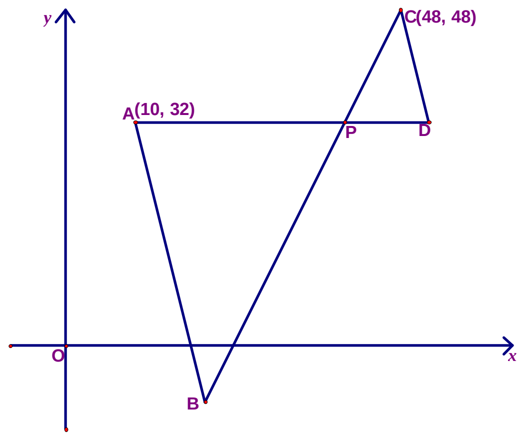
Example 17: In the diagram, A is the point (10, 32), C is the point (48, 48) and that $\vec{AB} = \frac{5}{2}\vec{CD}$ and $\vec{AB} = \begin{pmatrix} 15 \\ -40 \end{pmatrix}$.

(a) By means of a vector approach, find

(i) the coordinates of B, (ii) the coordinates of D, (iii) $\vec{BC}$.

(b) Given also APD and BPC are straight lines, find the coordinates of P.

(c) Find $\dfrac{\text{area of } \Delta PCD}{\text{area of } \Delta APB}$.



Solution:

(a)

$$\vec{OB} = \vec{AB} + \vec{OA}$$
$$= \begin{pmatrix} 25 \\ -8 \end{pmatrix}$$

$$\vec{OD} = \vec{CD} + \vec{OC} = \frac{2}{5}\begin{pmatrix} 15 \\ -40 \end{pmatrix} + \begin{pmatrix} 48 \\ 48 \end{pmatrix}$$
$$= \begin{pmatrix} 54 \\ 32 \end{pmatrix}$$

$$\vec{BC} = \vec{OC} - \vec{OB}$$
$$= \begin{pmatrix} 23 \\ 56 \end{pmatrix}$$

(b)

$$\text{ratio} = \frac{\sqrt{15^2 + 40^2}}{\sqrt{6^2 + 16^2}} = \frac{5}{2}$$

$$P = \left(10 + \frac{5}{7}(54 - 10), 32\right)$$
$$= \left(41\frac{3}{7}, 32\right)$$

(c)

$$\text{Area} = \left(\frac{2}{5}\right)^2$$
$$= \frac{4}{25}$$

Example 18: Find the angle between the vectors $\underset{\sim}{a} = 3\underset{\sim}{i} + \underset{\sim}{j}$ and $\underset{\sim}{b} = 2\underset{\sim}{i} - 3\underset{\sim}{j}$

Solution:

$$\cos\theta = \frac{\underset{\sim}{a}\cdot\underset{\sim}{b}}{|\underset{\sim}{a}||\underset{\sim}{b}|}$$

$$= \frac{\begin{pmatrix}3\\1\end{pmatrix}\cdot\begin{pmatrix}2\\-3\end{pmatrix}}{\left(\sqrt{3^2+1^2}\right)\left(\sqrt{2^2+(-3)^2}\right)}$$

$$= \frac{6-3}{\sqrt{10}\sqrt{13}}$$

$$= \frac{3}{\sqrt{130}}$$

$$\theta = 74.7°$$

Example 19:

(a) If $\underset{\sim}{a} \perp \underset{\sim}{b}$, show that $|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$

(b) Given that $\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{a}$, and $|\underset{\sim}{b}| = \sqrt{2}\,|\underset{\sim}{a}|$, show that $2\underset{\sim}{a}+\underset{\sim}{b}$ is perpendicular to $\underset{\sim}{b}$.

Solution:

(a)

$$|\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$|\underset{\sim}{a}-\underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + |\underset{\sim}{b}|^2$$

$$= |\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2 \quad \underset{\sim}{a}\cdot\underset{\sim}{b} = 0 \text{ since } \underset{\sim}{a}\perp\underset{\sim}{b}$$

$$\therefore |\underset{\sim}{a}+\underset{\sim}{b}|^2 = |\underset{\sim}{a}-\underset{\sim}{b}|^2$$

(b)

$$(\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{a} = 0 \quad \text{since } \underset{\sim}{a}+\underset{\sim}{b}\perp\underset{\sim}{a}$$

$$\underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{b}\cdot\underset{\sim}{a} = 0$$

$$\underset{\sim}{b}\cdot\underset{\sim}{a} = -|\underset{\sim}{a}|^2$$

$$(2\underset{\sim}{a}+\underset{\sim}{b})\cdot\underset{\sim}{b} = 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{b}\cdot\underset{\sim}{b}$$

$$= -2|\underset{\sim}{a}|^2 + |\underset{\sim}{b}|^2$$

$$= -2|\underset{\sim}{a}|^2 + 2|\underset{\sim}{a}|^2 \quad \left[\text{Given } |\underset{\sim}{b}| = \sqrt{2}\,|\underset{\sim}{a}|\right]$$

$$= 0$$

$$\therefore 2\underset{\sim}{a}+\underset{\sim}{b} \perp \underset{\sim}{b}$$

Example 20: If $3\underset{\sim}{i} + \lambda\underset{\sim}{j}$ and $2\lambda^2\underset{\sim}{i} - \underset{\sim}{j}$ are perpendicular vectors, find the value(s) of λ.

Solution:

$$(3\underset{\sim}{i} + \lambda\underset{\sim}{j})\cdot(2\lambda^2\underset{\sim}{i} - \underset{\sim}{j}) = 0$$

$$\begin{pmatrix}3\\\lambda\end{pmatrix}\cdot\begin{pmatrix}2\lambda^2\\-1\end{pmatrix} = 0$$

$$6\lambda^2 - \lambda = 0$$

$$\lambda = 0 \text{ or } \frac{1}{6}$$

Summary of Vectors

Equal Vectors: $\underset{\sim}{a} = \underset{\sim}{b} \iff |\underset{\sim}{a}| = |\underset{\sim}{b}|$ and $\underset{\sim}{a}, \underset{\sim}{b}$ are in the same direction

Parallel Vectors: $\underset{\sim}{a} \,/\!/\, \underset{\sim}{b} \iff \underset{\sim}{a} = \lambda \underset{\sim}{b}$ for some $\lambda \in \mathbb{R}, \lambda \neq 0$

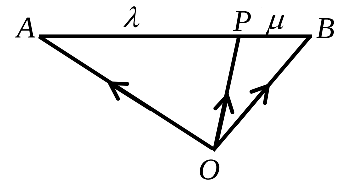
Collinear Points: A, B, C are collinear $\iff \overrightarrow{AB} = \lambda \overrightarrow{BC}$ for some $\lambda \in \mathbb{R}$

Ratio Theorem:

If P divides AB in the ratio $\lambda : \mu$, then $\overrightarrow{OP} = \dfrac{1}{\lambda + \mu}\left(\mu \overrightarrow{OA} + \lambda \overrightarrow{OB}\right)$

$(\text{i.e. } AP : PB = \lambda : \mu)$

$\left(\text{or } \dfrac{AP}{PB} = \dfrac{\lambda}{\mu}\right)$



If P is the mid-point of AB, then $\overrightarrow{OP} = \dfrac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OB}\right)$

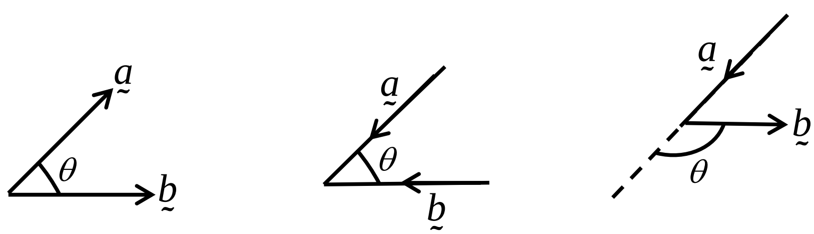
<u>2-Dimensional Vectors in Cartesian form</u>

Position Vector of a point P (relative to origin O) is $\underset{\sim}{r} = \overrightarrow{OP} = x\underset{\sim}{i} + y\underset{\sim}{j}$ or (x, y) or $\begin{pmatrix} x \\ y \end{pmatrix}$

Magnitude of $\underset{\sim}{r}$, $|\underset{\sim}{r}| = \sqrt{x^2 + y^2}$
Unit Vector of $\underset{\sim}{r}$, $\hat{\underset{\sim}{r}} = \dfrac{\underset{\sim}{r}}{|\underset{\sim}{r}|}$

*Scalar Product (Dot Product)

$\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta$ where $\theta = \measuredangle$ between $\underset{\sim}{a}$ & $\underset{\sim}{b}$



$\underset{\sim}{a} = x_1\underset{\sim}{i} + y_1\underset{\sim}{j}$ and $\underset{\sim}{b} = x_2\underset{\sim}{i} + y_2\underset{\sim}{j}$ then

$\underset{\sim}{a} \cdot \underset{\sim}{b} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = x_1x_2 + y_1y_2$

Impt. results:

(i) $\underset{\sim}{a} \cdot \underset{\sim}{a} = |\underset{\sim}{a}|^2 \cos 0 = a^2$

(ii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = 0 \iff a \perp b$

(iii) $\underset{\sim}{a} \cdot \underset{\sim}{b} = |\underset{\sim}{a}||\underset{\sim}{b}|\cos\theta = |\underset{\sim}{b}||\underset{\sim}{a}|\cos\theta = \underset{\sim}{b} \cdot \underset{\sim}{a}$

(iv) $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$

$(\underset{\sim}{b} + \underset{\sim}{c}) \cdot \underset{\sim}{a} = \underset{\sim}{b} \cdot \underset{\sim}{a} + \underset{\sim}{c} \cdot \underset{\sim}{a}$

(v) $|\underset{\sim}{a} + \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 + 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2$

$|\underset{\sim}{a} - \underset{\sim}{b}|^2 = |\underset{\sim}{a}|^2 - 2\underset{\sim}{a} \cdot \underset{\sim}{b} + |\underset{\sim}{b}|^2$