Topic: Further Trigonometry

The Compound Angle (Sum & Difference) Formulae

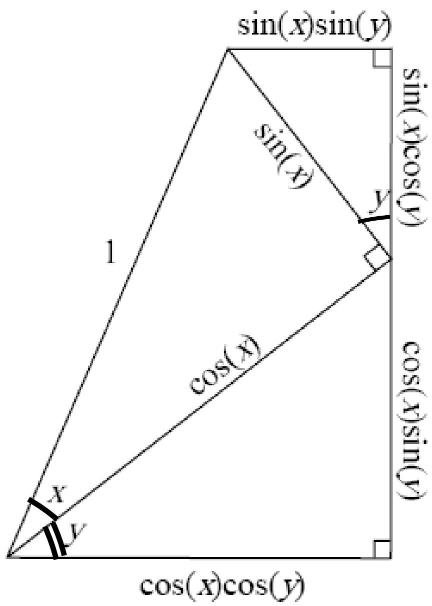


Diagram 1

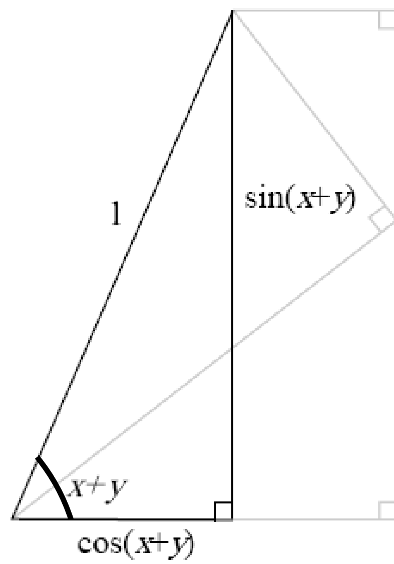


Diagram 2

Sum formulae:

Using both diagrams,

$$\begin{aligned} \sin(x+y) &= \sin(x)\cos(y)+\cos(x)\sin(y), \\ \cos(x+y) &= \cos(x)\cos(y)-\sin(x)\sin(y). \end{aligned}$$

Using the sum formulae for sin ($x + y$) and cos ($x + y$), we can derive the following:

$$\begin{aligned} \tan(x+y) &= \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin(x)\cos(y)+\cos(x)\sin(y)}{\cos(x)\cos(y)-\sin(x)\sin(y)} \\ &= \frac{\left(\frac{\sin(x)\cos(y)}{\cos(x)\cos(y)}\right)+\left(\frac{\cos(x)\sin(y)}{\cos(x)\cos(y)}\right)}{\left(\frac{\cos(x)\cos(y)}{\cos(x)\cos(y)}\right)-\left(\frac{\sin(x)\sin(y)}{\cos(x)\cos(y)}\right)} = \frac{\tan(x)+\tan(y)}{1-\tan(x)\tan(y)} \end{aligned}$$

$\sin(x-y) =$

$\cos(x-y) =$

$\tan(x-y) =$

Example 1: Find the value of $\sin 75°$ in surd form

Example 2: If $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$, calculate (without using tables or calculators)

(a) the value of $\sin(A+B)$ when A and B are both acute angles

(b) the value of $\cos(A+B)$ when A is obtuse and B is acute

Example 3: If A, B, C, D are all acute angles and $\tan A = \frac{1}{3}, \tan B = \frac{1}{5}, \tan C = \frac{1}{7}, \tan D = \frac{1}{8}$, evaluate $A + B + C + D$ without using calculators.

Example 4: If $\sin\theta = 2\sin(A-\theta)$, prove that $\tan\theta = \frac{2\sin A}{1+2\cos A}$. Given that $\cos A = \frac{1}{3}$, where A is an acute angle, solve the equation $\sin\theta = 2\sin(A-\theta)$ for $0° \le \theta \le 360°$.

Example 5: Solve the equations for values of x where $0° \le x \le 360°$.

(a) $\cos 3x \cos x + \sin 3x \sin x = \frac{1}{2}$,

(b) $(\sin x + \cos x)^2 = 2\sin(45° + x)\sin(45° - x)$.

Exercise

Marshall Cavendish TB Pg 330 – 332 Ex 13.1 Q3, 8, 10, 13, 15, 17, 18, 20

Challenge: H.O.T Corner Q22, Q23

Q23 A triangle has two acute angles, A and B. Show that triangle is right-angled if and only if, $\sin^2 A + \sin^2 B = \sin(A+B)$

Double Angle Formulae

The Proof:

i) $\sin(A + A) = \sin 2A =$

ii) $\cos(A + A) = \cos 2A =$

iii) $\tan(A + A) = \tan 2A =$

Variations of the Double Angle formulas

The double angle formulas are used to relate ***ANY*** angle and its half angle.

For example:

a) $\sin 4x =$

b) $\cos 6y =$

c) $\sin A =$

d) $\cos A =$

e) $\tan A =$

Therefore, in general: For all integral values of *n*, (usually *n* is even)

$$\sin nA = 2\sin\frac{n}{2}A\cos\frac{n}{2}A$$

$$\begin{aligned}\cos nA &= \cos^2\frac{n}{2}A - \sin^2\frac{n}{2}A\\ &= 1 - 2\sin^2\frac{n}{2}A\\ &= 2\cos^2\frac{n}{2}A - 1\end{aligned}$$

$$\tan nA = \frac{2\tan\frac{n}{2}A}{1-\tan^2\frac{n}{2}A}$$

Example 1: Express $\sin 3x$, $\cos 3x$ and $\tan 3x$ in terms of *x*.

Example 2: Given that $\sin A = \frac{3}{5}$ where $0^\circ < A < 90^\circ$, find the value of $\sin 2A$

Example 3: If $\cos\theta = -\frac{7}{9}$ $(90^\circ < \theta < 180^\circ)$, find the value of $\tan\frac{\theta}{2}$.

Example 4: Prove the identities

(a) $\frac{\sin 3x}{\cos x} + \frac{\cos 3x}{\sin x} = 2\cot 2x$

(b) $\frac{1-\cos\theta}{1+\cos\theta} \equiv \tan^2\left(\frac{\theta}{2}\right)$

(c) $\cos 4\theta = 4(\cos^4\theta + \sin^4\theta) - 3$

Example 5: Solve the following equations

(a) $\cos 2\theta = 3\cos\theta - 2$ for $0^\circ \le \theta \le 360^\circ$

(b) $\sin\frac{\theta}{2} = \cos\theta$ for $-90^\circ < \theta < 360^\circ$

(c) $\sin 4x + \cos 2x = 0$ for $0° \le x \le 360°$

Exercise

Marshall Cavendish TB Pg 336 – 337 Ex 13.2 Q1, 8, 9, 10, 11, 15, 16, 17,

Challenge: H.O.T Corner Q19, Q20

Q20 Suppose *A* and *B* are two angles such that $\sin A + \sin B = 1$ and $\cos A + \cos B = 0$.

Find the value of $12\cos 2A + 4\cos 2B$

R- Formulae

This formulae is used primarily to combine Trigo Functions. One of its applications is to solve trigo equations of the form 3 cos x – 4 sin x = 1.

Example 0: $2\cos(\theta - 60^\circ) =$

For $a > 0$, $b > 0$ and α is acute,

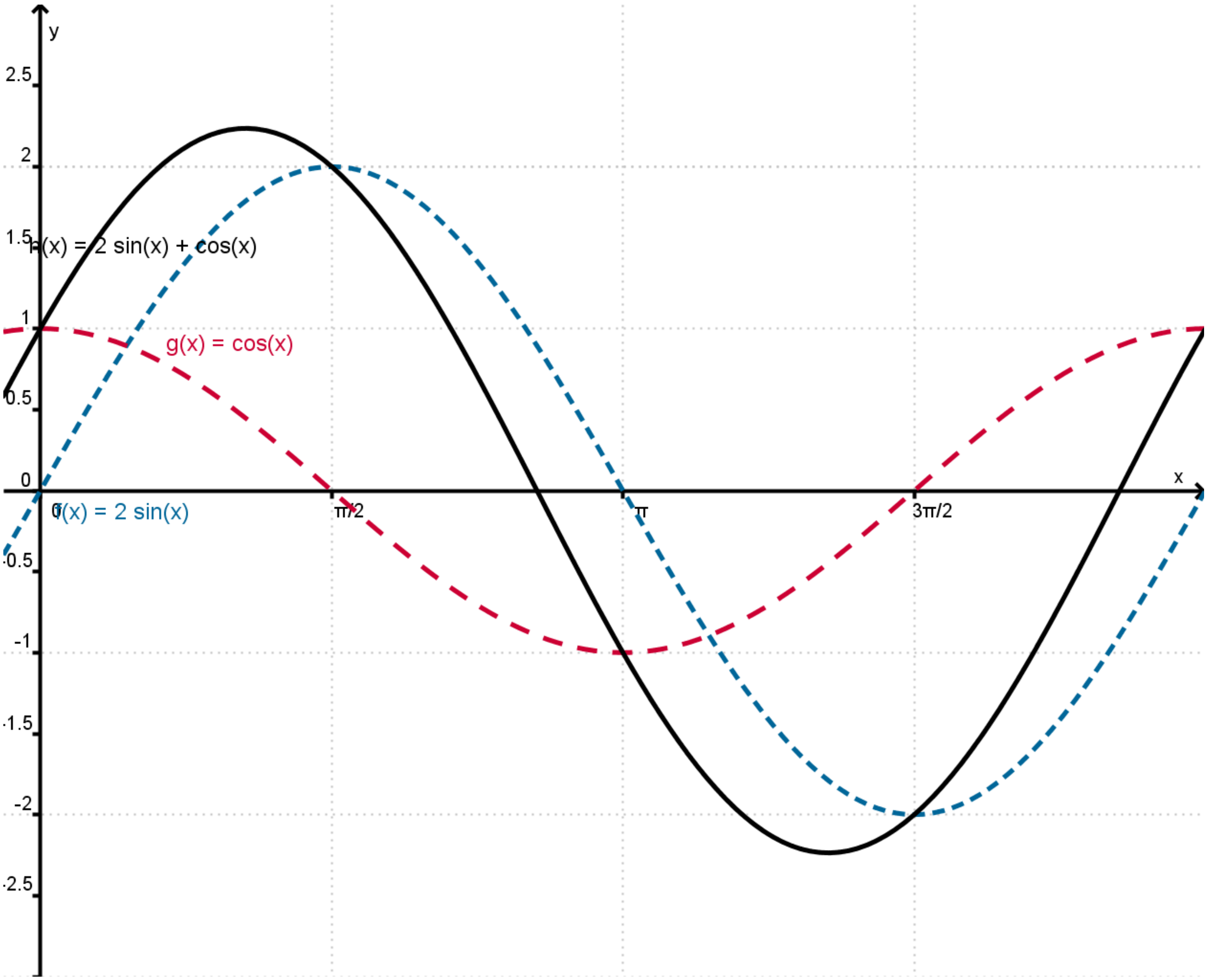
$$a\cos\theta \pm b\sin\theta = R\cos(\theta \mp \alpha)$$
$$a\sin\theta \pm b\cos\theta = R\sin(\theta \pm \alpha)$$

Where

Mathematical Proof:

(1) $a\cos\theta + b\sin\theta = R\cos(\theta - \alpha)$

Graphical Representation:

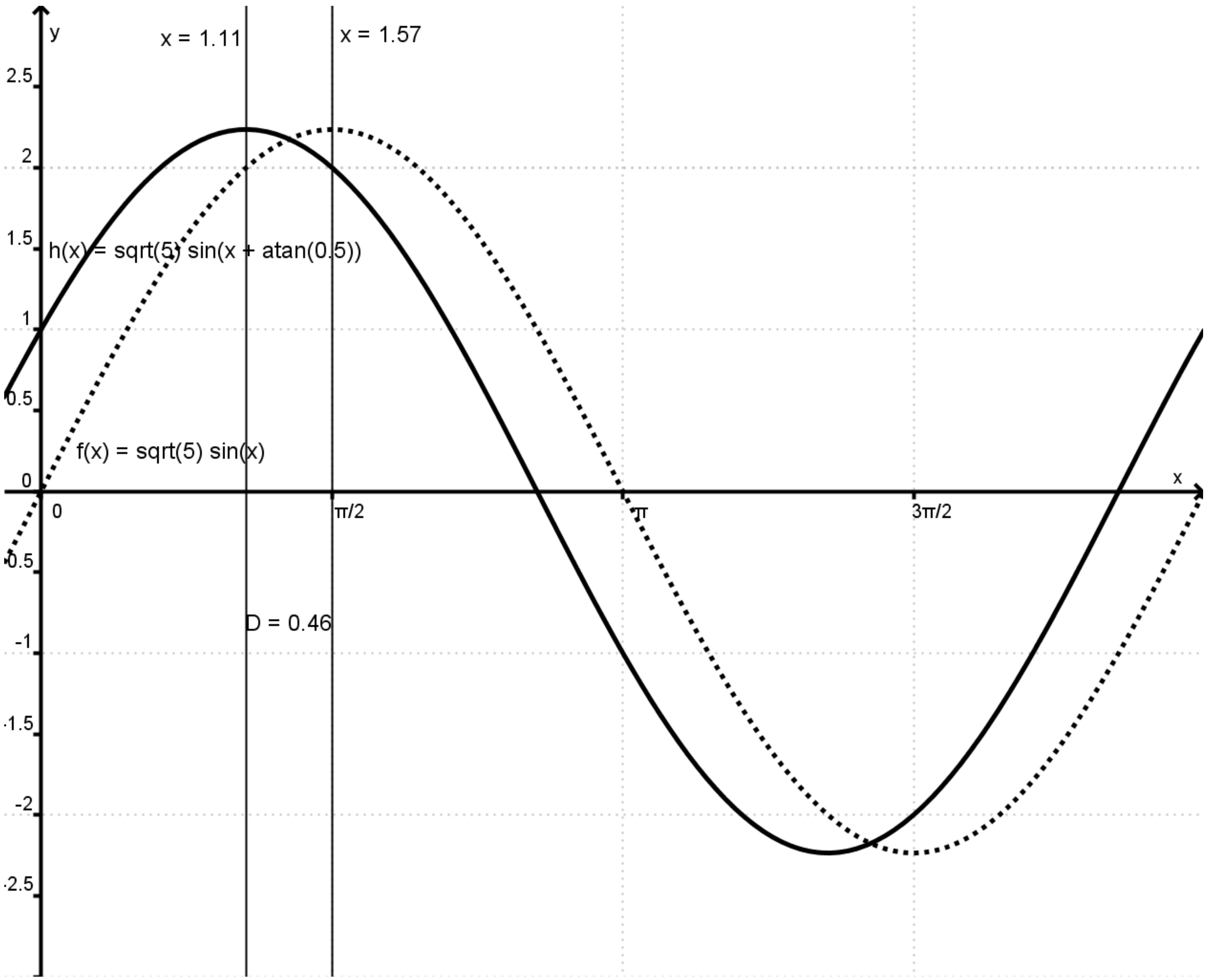


The graph of $y = 2\sin x + \cos x$ can also be represented by

Where
maximum value of y = _______, occurs when x = ________

minimum value of y = _______, occurs when x = ________

Understanding Amplitude, frequency and phase shift:



$y = \sqrt{5}\sin(x + 0.46)$ is the graph of $y = \sqrt{5}\sin x$ shifted horizontally (phase shift) to the left by an angle of 0.46^c. There is no change in the amplitude nor the frequency.

Likewise, $y = \sqrt{5}\sin(x - 0.46)$ is simply the graph of $y = \sqrt{5}\sin x$ shifted to the right by 0.46^c.

Example 1: Express $5\sin\theta + 12\cos\theta$ in the form

(a) $R\sin(\theta+\alpha)$ where $R > 0$ and α is acute.

(b) $R\cos(\theta-\alpha)$ where $R > 0$ and α is acute.

Example 2: Find all angles between 0° and 360° which satisfy the equation $5\sin\theta + 12\cos\theta = 10$.

Example 3: Find the maximum and minimum values of $5\sin\theta + 12\cos\theta$, and the corresponding values of *x*, where $0 \le x \le 2\pi$

Example 4: Solve the following equations for $0^\circ \le \theta \le 360^\circ$

(a) $\sqrt{3}\sin\frac{\theta}{2} + \cos\frac{\theta}{2} = 1$

(b) $2\cot\theta = 3 + 2\ \mathrm{cosec}\ \theta$

(c) $8\cos^2\theta - 12\cos\theta\sin\theta = 1$

Example 5: Given that $\mathrm{f}(x) = 10\cos^2 x + 2\sin^2 x + 6\sin x\cos x$, express $\mathrm{f}(x)$ in the form $A + B\cos(2x-\alpha)$, where *A*, *B* are constants and $\tan\alpha = \frac{3}{4}$. Hence, by finding the greatest and least values of $\mathrm{f}(x)$, sketch the graph of $\mathrm{f}(x)$ for $0° \le x \le 360°$

Exercise

Marshall Cavendish TB Pg 341 – 343 Ex 13.3 Q5, 8, 10, 13, 16, 17

Challenge: H.O.T Corner Q19, Q20

Q20 Find the range of the function $y = (\sin x + 3\sqrt{2})(\cos x + 3\sqrt{2})$

Factor Formulae

Factor Formulae can be used to combine the sum and difference of sine and cosine of **two angles**.

Factor formulae can be derived from compound angle formulae as follows:

$$\sin(A+B) = \sin A \cos B + \cos A \sin B \to (1)$$

$$\sin(A-B) = \sin A \cos B - \cos A \sin B \to (2)$$

(1) + (2):

$$\sin(A+B) + \sin(A-B) =$$

(1) – (2):

$$\sin(A+B) - \sin(A-B) =$$

In other words,

$$\sin X + \sin Y =$$

$$\sin X - \sin Y =$$

Similarly,

$$\cos(A+B) = \cos A \cos B - \sin A \sin B \to (3)$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B \to (4)$$

(3) + (4):

$$\cos(A+B) + \cos(A-B) =$$

(3) – (4):

$$\cos(A+B) - \cos(A-B) =$$

In other words,

$$\cos X + \cos Y =$$

$$\cos X - \cos Y =$$

Example 1: Prove the following identities

(a) $\cos x + \cos 3x + \cos 5x + \cos 7x = 4\cos x \cos 2x \cos 4x$

(b) $\dfrac{\sin A + \sin 2A + \sin 4A + \sin 5A}{\cos A + \cos 2A + \cos 4A + \cos 5A} = \tan 3A$

(c) $\dfrac{\sin 2A + \sin 4A}{\cos 2A - \cos 4A} = \cot A$

Example 2: Solve the following equations for values of x from 0^{o} to 360^{o} inclusive

(a) $\cos x + \cos 5x = 0$

(b) $\sin(x + 10^\circ) + \sin x = 0$

(c) $\sin(x + 30^\circ) + \sin(x - 30^\circ) = \dfrac{\sqrt{3}}{2}$

(d) $\cos 3x + \sin 2x = 0$

(e) $\sin(x + 15^\circ)\cos(x - 15^\circ) = 1$

Exercise

Q1 Prove the following identities

(a) $\dfrac{\cos(2A - 3B) + \cos 3B}{\sin(2A - 3B) + \sin 3B} \equiv \cot A$

(b) $\sin(45^\circ + A)\cos(45^\circ - A) + \cos(45^\circ + A)\sin(45^\circ - A) \equiv 1$

(c) $\dfrac{\sin x - \sin 2x + \sin 3x}{\cos x - \cos 2x + \cos 3x} \equiv \tan 2x$

(d) $\cos(90^\circ + x) + \cos(90^\circ - x) \equiv 0$

Q2 Solve the following equations, for values of x from 0^{o} to 360^{o} inclusive:

(a) $\cos 4x - \cos x = 0$

(b) $\cos(2x + 10^\circ) + \cos(2x - 10^\circ) = 0$

(c) $\cos(x + 120^\circ) + \cos x = 1$

(d) $\cos(x + 60^\circ)\cos(x - 60^\circ) = \dfrac{1}{4}$

Q3 Prove that $1+\cos 2\theta+\cos 4\theta+\cos 6\theta=4\cos\theta\cos 2\theta\cos 3\theta$.

Hence find all the angles θ between $0°$ and $360°$ inclusive which satisfy the equation

$$\cos 2\theta+\cos 4\theta+\cos 6\theta=-1$$

Challenge:

Q4 Find the value of $\dfrac{1}{\sin 10°}-4\sin 70°$.

Q5 If A, B, C are the interior angles of a triangle, show that

(a) $\sin A+\sin B+\sin C\equiv 4\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}$

(b) $\sin 2A+\sin 2B+\sin 2C\equiv 4\sin A\sin B\sin C$

(c) $\sin 4A+\sin 4B+\sin 4C\equiv -4\sin 2A\sin 2B\sin 2C$