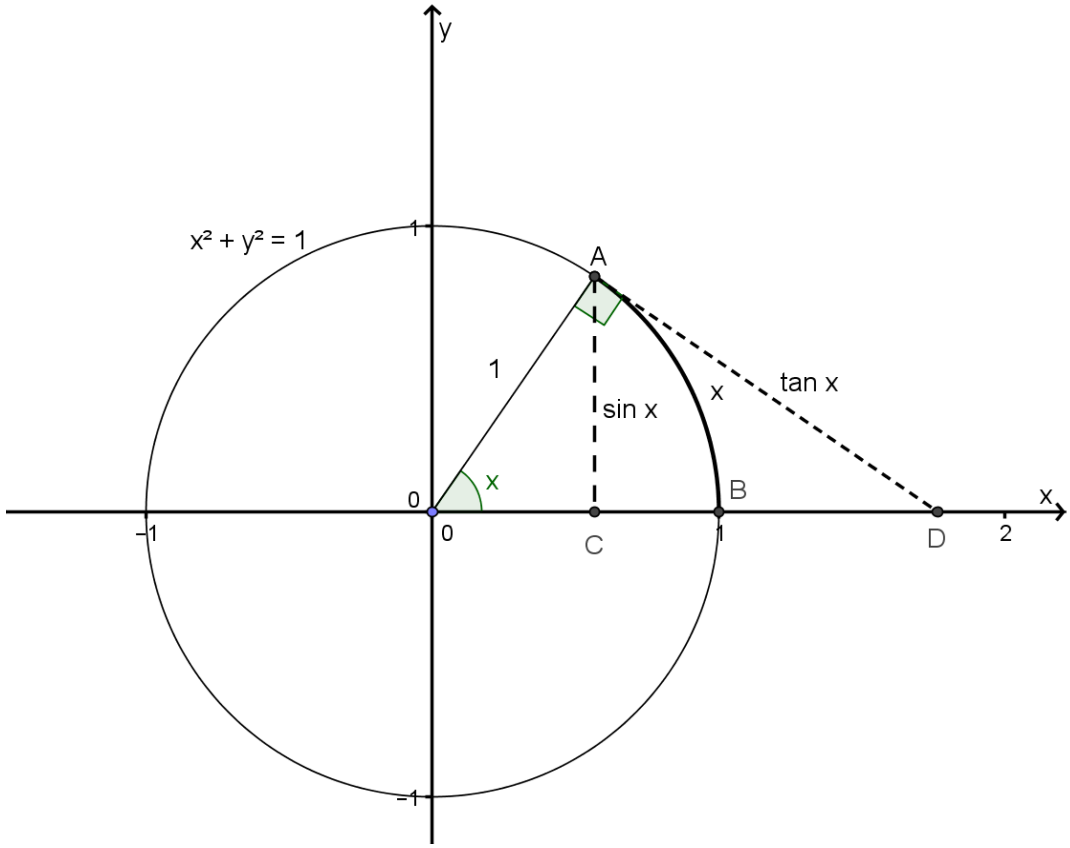
Differentiation of Trigonometric Functions

Consider the diagram below which shows part of a unit circle:



A is a point on the circle. *OA* makes an angle of *x*° with the positive *x*-axis.

Using circular measure, arc *AB* = *x* units

Using pythagoras theorem, *AC* = sin *x* and *AD* = tan *x*

Comparing the lengths of *AC*, *AB* and *AD*,

For $0 < x < \frac{\pi}{2}$,

$$\sin x < x < \tan x$$

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$$

$$\frac{\sin x}{\sin x} < \frac{x}{\sin x} < \frac{\sin x}{\cos x \sin x}$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$$

$$1 > \frac{\sin x}{x} > \cos x$$

Since $\lim_{x \to 0} 1 = 1$ and $\lim_{x \to 0} \cos x = 1$, $\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h}\\ &= \lim_{h\to 0}\frac{\sin x\left(\cos h-1\right)+\cos x\sin h}{h}\\ &= \lim_{h\to 0}\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\end{aligned}$$

Now,

$$\begin{aligned}\lim_{h\to 0}\frac{\cos h-1}{h} &= \lim_{h\to 0}\frac{\cos h-1}{h}\cdot\frac{\cos h+1}{\cos h+1}\\ &= \lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}\\ &= \lim_{h\to 0}\frac{-\sin^2 h}{h(\cos h+1)}\\ &= \lim_{h\to 0}\left(-\sin h\right)\cdot\left(\frac{\sin h}{h}\right)\cdot\left(\frac{1}{\cos h+1}\right)\\ &= 0\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\sin x &= \lim_{h\to 0}\sin x\cdot\frac{\cos h-1}{h}+\cos x\cdot\frac{\sin h}{h}\\ &= \sin x\cdot 0+\cos x\cdot 1\\ &= \cos x\end{aligned}$$

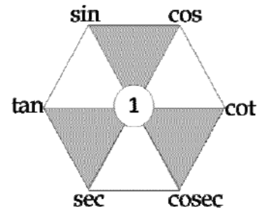
Similarly,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\cos x &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\cos h-\sin x\sin h-\cos x}{h}\\ &= \lim_{h\to 0}\frac{\cos x\left(\cos h-1\right)-\sin x\sin h}{h}\\ &= \lim_{h\to 0}\cos x\frac{\cos h-1}{h}-\sin x\cdot\frac{\sin h}{h}\\ &= 1\cdot 0-\sin x\cdot 1\\ &= -\sin x\end{aligned}$$

And

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\tan x &= \frac{\mathrm{d}}{\mathrm{d}x}\frac{\sin x}{\cos x}\\ &= \frac{\cos x\left(\cos x\right)-\sin x\left(-\sin x\right)}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

Summary



(i) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$ (Note the ***NEGATIVE*** sign)

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan x) = \sec^2 x$

Example 1 Differentiate the following expressions with respect to x:

(a) $2+3\sin x$ (b) $(3+\tan x)^3$ (c) $\frac{\cos x}{x}$ (d) $\sin x \tan x$ (e) $x^2 \sin x$

Example 2 Find the derivative for $\sec x$, $\operatorname{cosec} x$ and $\cot x$

1) $\frac{\mathrm{d}}{\mathrm{d}x}\sec x =$

2) $\frac{\mathrm{d}}{\mathrm{d}x}\operatorname{cosec} x =$

3) $\frac{\mathrm{d}}{\mathrm{d}x}\cot x =$

Differentiation of sin (*ax* + *b*), cos (*ax* + *b*) and tan (*ax* + *b*)

This is ***FUNCTION of FUNCTION*** so we use chain rule:

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\sin(ax+b) = a\cos(ax+b)$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\cos(ax+b) = -a\sin(ax+b)$ (Note the ***NEGATIVE*** sign)

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}\tan(ax+b) = a\sec^2(ax+b)$

Example 3 Differentiate the following expressions with respect to x

(a) $\cos 4x$ (b) $\sin x^2$ (c) $x\left(\cos 2x - \sin x\right)$ (d) $\sin 2x\cos 3x$

Differentiation of sin*ⁿx*, cos*ⁿx* and tan*ⁿx*

This is also ***FUNCTION of FUNCTION***. Use Chain Rule as well.

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\sin^n x = n\sin^{n-1} x.\cos x$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\cos^n x = -n\cos^{n-1} x.\sin x$ (Note the ***NEGATIVE*** sign)

(iii) $\frac{\mathrm{d}}{\mathrm{d}x}\tan^n x = n\tan^{n-1} x.\sec^2 x$

Example 4 Differentiate the following expressions with respect to x:

(a) $\tan^3 x$ (b) $\sin^3\left(2x+1\right)$ (c) $\cos^4 3x$

(Note that (b) and (c) are **function of function of function**)

In general if we have function of function of function, **the "outside to inside" approach applies, as follows:**

$$\frac{d}{dx}\sin^n(ax+b) = n\left[\sin(ax+b)\right]^{n-1}.\cos(ax+b).a$$

Sequence: power first ↓ angle last ↙

function second ⟶ $\sin^n(ax+b)$

NOTE:

All formulae in calculus for trigonometric functions are only true for radian measure. Angles in degrees must be converted to radian.

Example 5 Differentiate the following expressions with respect to x

(a) $\sqrt{3+2\cos 2x}$ (b) $\sqrt{1+2\sin^2 3x}$

(Note that (b) is a **function of function of function of function**)

Example 6 Given that $y=\sin 3x+\cos^3 x$, find the value of $\frac{dy}{dx}$ when $x=\frac{\pi}{4}$.

Example 7 Find the value of k for which $\frac{d(2x-\sin 2x)}{dx}=k\sin^2 x$.

Example 8 Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for each of the following functions:

(a) $y = \cot^2(3x+6)^5$ (b) $y = (2x^4+1)^3\cos^2 x$ (c) $y = \cos^4 2x\tan^2 x$

Example 9 (Tangent and normal)

Find the equation of the tangent and the normal to the curve $y = 1+\cos x$ at the point where $x = \frac{\pi}{6}$.

Show that the area enclosed by the tangent, the normal and the x-axis is $\frac{5}{4}\left(1+\frac{\sqrt{3}}{2}\right)^2$ units^2.

Example 10 (Rate of change)

Given that x changes at a constant rate of $\frac{\pi}{30}$ radians per second, find the rate of change of $\sqrt{1+\sin x}$ when $x = \frac{\pi}{3}$.

Example 11 (Stationary point)

Find the coordinates of the turning points on the curve $y = \sin x \cos^2 x$ for $0 < x < \pi$.

Example 12 (Max/Min)

In the figure, a circle with centre *O* and radius *r* circumscribes an isosceles triangle *ABC* with *AB* = *AC*. If $\angle BAC = \theta$, where θ is acute, show that the area of triangle *ABC*, is given by $A = r^2 \sin\theta(1+\cos\theta)$.

Hence show that *A* is maximum when $\theta = \dfrac{\pi}{3}$.