Differentiation of Exponential Functions

What is *e*?

The constant **2.718281828459045…** is designated as ***e***.

It is named after the famous mathematician Leonhard Euler (1707-1783) as ***Euler's*** **constant**. It is an irrational number, and is defined as the limiting value of $\left(1+\frac{1}{n}\right)^n$ as *n* approaches infinity.

$$e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$$

n	$\left(1+\frac{1}{n}\right)^n$
10	2.59374246…
10^2	2.704813829…
10^3	2.716923932…
10^4	2.718145927…
⋮	⋮
⋮	⋮
10^7	2.718281693…
10^8	2.718281815…

By 1st principles,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^x &= \lim_{h\to0}\frac{e^{x+h}-e^x}{h}\\ &= \lim_{h\to0}\frac{e^x\left(e^h-1\right)}{h}\end{aligned}$$

Now, from the definition of *e*, we have $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

Let $n=\frac{1}{h}$ so when $n\to\infty, h\to0$ and we can write

$$e=\lim_{h\to0}\left(1+h\right)^{\frac{1}{h}}$$

So for small values of *h*, $e\approx\left(1+h\right)^{\frac{1}{h}}$

$$\therefore e^h\approx\left(1+h\right)$$

Hence, $$\begin{aligned}\lim_{h\to0}\frac{\left(e^h-1\right)}{h} &= \lim_{h\to0}\frac{\left(1+h-1\right)}{h}\\ &= 1\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}e^x &= \lim_{h\to0}e^x\cdot\frac{\left(e^h-1\right)}{h}\\ &= e^x\cdot1\\ &= e^x\end{aligned}$$

*Similarly, for exponential functions of other base, a:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x} a^x &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\
&= \lim_{h\to 0} \frac{a^x\left(a^h - 1\right)}{h} \\
&= \lim_{h\to 0} a^x \cdot \frac{\left(a^h - 1\right)}{h}
\end{aligned}$$

$$\begin{aligned}
\lim_{h\to 0} \frac{\left(a^h - 1\right)}{h} &= \lim_{h\to 0} \frac{\left(e^{\ln a^h} - 1\right)}{h} \\
&= \lim_{h\to 0} \frac{\left(e^{h\cdot \ln a} - 1\right)}{h} \\
&= \frac{\ln a}{\ln a} \cdot \lim_{h\to 0} \frac{\left(e^{h\cdot \ln a} - 1\right)}{h} \\
&= \ln a \lim_{h+\ln a\to 0} \frac{\left(e^{h\cdot \ln a} - 1\right)}{h\cdot \ln a} \\
&= \ln a \cdot 1 \\
&= \ln a
\end{aligned}$$

Hence, $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$

Summary

(i) $\frac{\mathrm{d}}{\mathrm{d}x} e^x = e^x$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x} e^{ax+b} = ae^{ax+b}$

*(iii) $\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x \ln a$

*(iv) $\frac{\mathrm{d}}{\mathrm{d}x} a^{bx+c} = ba^{bx+c} \ln a$

In other words, "copy exactly the same exponential function, then multiply with the derivative of its power":

$$\frac{\mathrm{d}}{\mathrm{d}x} e^{\mathrm{f}(x)} = e^{\mathrm{f}(x)} \cdot \mathrm{f}'(x)$$

Example 1 Differentiate the following expressions with respect to x:

(a) e^{3x} (b) $e^{\sin 2x}$ (c) xe^{-2x} (d) e^{2x+1} (e) $e^x \sin x + e$

Exercise 1

Find the derivative of the following with respect to x

(a) $\frac{\mathrm{d}}{\mathrm{d}x} 10e^{3-5x}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x} e^{2x^2+3x+7}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x} 2e^{3x} + 4e^{-5x}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{e^{2x+3}}{e^{x}}$

(e) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{3\left(e^{2x}\right)^3}{e^{7x-2}}$

(f) $\frac{\mathrm{d}}{\mathrm{d}x} \left(e^{1-x}\right)^3$

(g) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{e^{3x}+e^{-2x}+e^{x}}{e^{x}}$

(h) $\frac{\mathrm{d}}{\mathrm{d}x} \left(e^{x}+e^{2x}\right)^4$

(i) $\frac{\mathrm{d}}{\mathrm{d}x} x^2 e^{3x}$

(j) $\frac{\mathrm{d}}{\mathrm{d}x} \frac{e^{x}}{\sin x}$

Differentiation of Natural Logarithm Functions

By first principles,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \lim_{h\to 0}\frac{\ln(x+h)-\ln x}{h}$$

$$= \lim_{h\to 0}\frac{\ln\frac{x+h}{x}}{h} \qquad \left[\ln p - \ln q = \ln\frac{p}{q}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x}\right)$$

$$= \lim_{h\to 0}\ln\left(1+\frac{h}{x}\right)^{\frac{1}{h}} \qquad \left[r\ln p = \ln p^{r}\right]$$

$$= \lim_{h\to 0}\ln\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}\cdot\frac{1}{x}}$$

$$= \lim_{h\to 0}\ln\left(\left(1+\frac{1}{\frac{x}{h}}\right)^{\frac{x}{h}}\right)^{\frac{1}{x}}$$

$$= \lim_{h\to 0}\ln(e)^{\frac{1}{x}} \qquad \text{since } e = \left[1+\frac{1}{n}\right]^{n}$$

$$= \lim_{h\to 0}\frac{1}{x}\ln e$$

$$= \frac{1}{x}$$

Alternatively,

Let $y = \ln x$

$$\therefore x = e^{y}$$

$$\frac{\mathrm{d}x}{\mathrm{d}y} = e^{y}$$

Hence

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{e^{y}}$$

$$= \frac{1}{x}$$

We can also use the previous result to derive the derivative of e^x :

Let $y = e^x$

$$\ln y = \ln e^x$$
$$\ln y = x \ln e$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y$$
$$= e^x$$

Likewise, we can do the same for any exponential function of any base a^x

Let $y = a^x$

$$\ln y = \ln a^x$$
$$\ln y = x \ln a$$
$$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = \ln a$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \ln a$$
$$= a^x \cdot \ln a$$

***Differentiation of other Logarithm Functions**

Similarly,

$$y = \log_a x, a > 0, a \neq 1$$
$$\therefore x = a^y$$
$$\frac{\mathrm{d}x}{\mathrm{d}y} = a^y \cdot \ln a \qquad \text{[From above result]}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{a^y \cdot \ln a}$$
$$= \frac{1}{x \cdot \ln a}$$

Summary

(i) $\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$

(ii) $\frac{\mathrm{d}}{\mathrm{d}x}\ln(ax+b) = \frac{1}{ax+b}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(ax+b) = \frac{a}{ax+b}$

*(iii) $\frac{\mathrm{d}}{\mathrm{d}x}\log_a x = \frac{1}{x\ln a}$

*(iv) $\frac{\mathrm{d}}{\mathrm{d}x}\log_a(bx+c) = \frac{1}{(bx+c)\cdot\ln a}\cdot\frac{\mathrm{d}}{\mathrm{d}x}(bx+c) = \frac{b}{(bx+c)\cdot\ln a}$

In other words, take the reciprocal of “ the same exact inside function”, then multiply with the derivative of the inside function”,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln \mathrm{f}(x) = \frac{1}{\mathrm{f}(x)}\cdot \mathrm{f}'(x)$$

How would you differentiate $\ln\left[f(x)^m g(x)^n\right]$ and $\ln\left[\frac{f(x)^m}{g(x)^n}\right]$?

Example 2 Differentiate the following expressions with respect to *x*:

(a) $\ln\sqrt{5-2x}$ (b) $\ln x \ln 3x$ (c) $\ln\left(\frac{1-x}{3+x}\right)$

Exercise 2 Differentiate the following with respect to *x*

(a) $6\ln(9-2x)$

(b) $\ln(9x+2)^3$

(c) $\ln\frac{4}{5-3x}$

(d) $\ln\frac{(2-x)^2}{(5x+1)^3}$

(e) $\ln\sqrt{\dfrac{5+2x}{4-3x}}$

(f) $\ln\left(x^2\right)+\left(\ln x\right)^2$

(g) $\dfrac{\ln x}{x}$

(h) $x\ln x$

(i) $x^3\ln\left(\sin^3 x\right)$

(j) $\ln\sqrt{\dfrac{(1-2x)e^{-2x}}{2x+1}}$

Example 2 Given that $y = Ae^{kx}$ and $\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 4e^{2x}$ find the values of constants *A* and *k*.

Example 3 Find the equation of the normal to curve $y = e^{2x}$ at the point where x=1, in terms of *e*.

Example 4 (a) Differentiate $x\ln(3x-1)^2$ with respect to *x*.

(b) Given $y = e^x \cos 2x$, find $\frac{dy}{dx}$ and hence determine, for $0 \le x \le \pi$, the values of *x* for which *y* is stationary.

Example 5 Given that a curve has the equation $y = 10(x+2)e^x$.

(i) Find the coordinates of the stationary point on the curve.

(ii) State the coordinates of the points at which the curve crosses the *x* and *y* axes.

(iii) Sketch the curve for $-4 \le x \le 1$.