Sec 4 Integrated Mathematics : Vectors (Self-Study Topic)

Name : ____________________ (   ) Sec 4 ____   Date :__________

(A) Scalar Versus Vector

All physical quantities can be classified into scalar or vector.

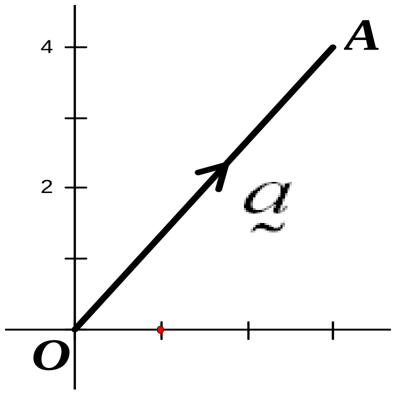
A ***scalar*** quantity possesses only ***magnitude***.
e.g. mass, distance, speed, work, energy time, temperature, length, area, volume.

A ***vector*** quantity possesses both ***magnitude*** and ***direction***.
e.g. force, displacement, velocity, acceleration, momentum, moment.

(B) Representation of Vector

(i) Column Vector

Geometrically, a vector can be represented by a ***directed line segment***. *The arrow indicates its direction while the length is proportional to the vector's magnitude.*



The diagram shows a point O being displaced to a point A,
The displacement can be represent by several notations

such as $\overrightarrow{OA}$, **a** or $\underset{\sim}{a}$.

If O is the origin and the coordinates of A is (3, 4), then $\overrightarrow{OA}$ can be expressed as a ***column vector*** , $\underset{\sim}{a} = \overrightarrow{OA} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

> If X has coordinates (m, n) then $\overrightarrow{OX} = \begin{pmatrix} m \\ n \end{pmatrix}$, conversely if $\overrightarrow{OY} = \begin{pmatrix} p \\ q \end{pmatrix}$ then the coordinates of Y is (p, q).

(ii) Magnitude of a vector

The ***magnitude*** of $\overrightarrow{OA}$ is denoted by $|\overline{OA}|$ or $|\underset{\sim}{a}|$. Using *Pythagoras Theorem*, $|\overline{OA}| = \sqrt{3^2 + 4^2} = 5$.

> *For any vector with column vector* $\begin{pmatrix} p \\ q \end{pmatrix}$, *its magnitude is given by* $\sqrt{p^2 + q^2}$.

(C) Equal Vectors, Negative Vectors, Zero (or null) Vectors

(i) If two vectors have the ***same magnitude*** and the ***same direction***, they are ***equal***.

(ii) ***Negative vectors*** have the same magnitude but opposite directions.
e.g. $\overrightarrow{AB}$ and $\overrightarrow{BA}$ are ***negative vectors*** of each other and we write $\overrightarrow{AB} = -\overrightarrow{BA}$.

(iii) $\overrightarrow{AB} - \overrightarrow{AB} = \mathbf{0}$, **0** is a ***zero vector*** . It has *zero magnitude and its direction is indeterminate*.

Example 1

State whether the following statement is true or false.

(i) $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right| \Rightarrow \overrightarrow{AB} = \overrightarrow{CD}$, **(ii)** $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow \left|\overrightarrow{AB}\right| = \left|\overrightarrow{CD}\right|$.

Answers: (i) False, though AB has the same magnitude as CD, they may not have the same direction.
(ii) True, if two vectors are equal, then both their magnitude and their direction must be the same.

Example 2

Given that $ABCD$ is a square,

(i) then $\overrightarrow{AB} = \overrightarrow{BC}$? **(ii)** then $\overrightarrow{AD} = \overrightarrow{BC}$? **(iii)** then $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{BC}\right|$?

Answers: (i) False, different directions (ii) True (iii) True

Example 3

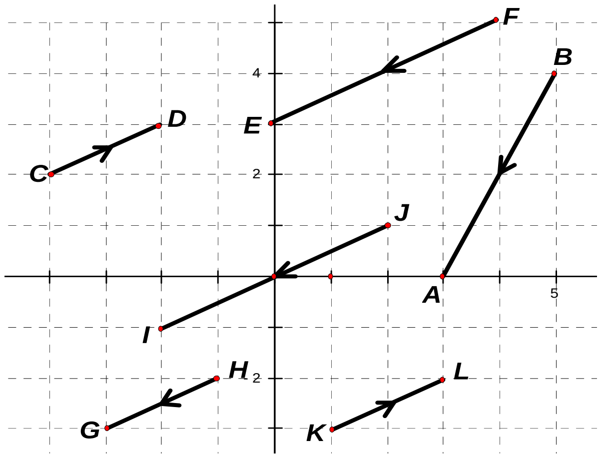
State whether the following statement is true or false.

(i) $\overrightarrow{AB} = \overrightarrow{EF}$,

(ii) $\left|\overrightarrow{AB}\right| = \left|\overrightarrow{EF}\right| = \left|\overrightarrow{IJ}\right|$,

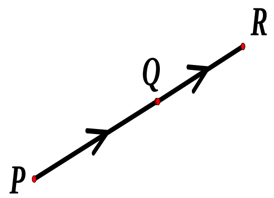
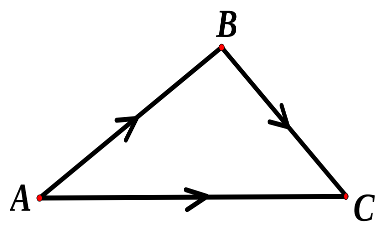
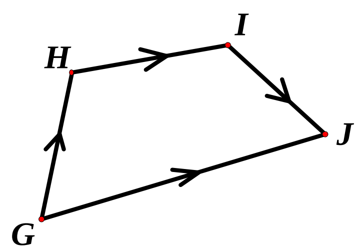
(iii) $\overrightarrow{CD} = \overrightarrow{KL} = -\overrightarrow{HG}$.

Answers : (i) False (ii) True (iii) True



(D) Addition of Coplanar Vectors

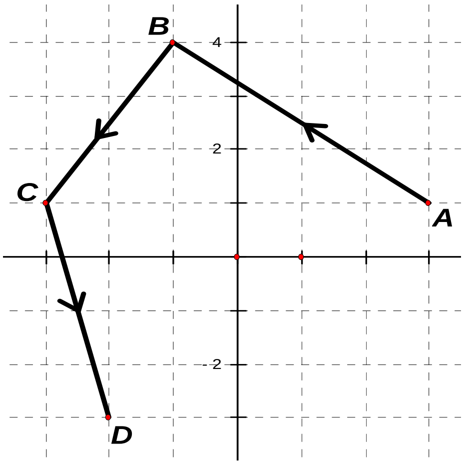
Coplanar vectors are vectors lying on the same plane.

Case 1 : *Addition of two vectors that are **parallel**.* $\overrightarrow{PQ} + \overrightarrow{QR} = \overrightarrow{PR}$ *Adding vectors* $\overrightarrow{PQ}$ *and* $\overrightarrow{QR}$ *give rise to **resultant vector*** $\overrightarrow{PR}$. *We can write* $\left\|\overrightarrow{PQ}\right\| + \left\|\overrightarrow{QR}\right\| = \left\|\overrightarrow{PR}\right\|$.	
Case 2 : *Addition of two vectors that are **non-parallel**, we write* $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ *Note that here* $\left\|\overrightarrow{AB}\right\| + \left\|\overrightarrow{BC}\right\| \neq \left\|\overrightarrow{AC}\right\|$.	
Case 3 : *Addition of more than two non-parallel vectors* $\overrightarrow{GH} + \overrightarrow{HI} + \overrightarrow{IJ} = \overrightarrow{GJ}$ *Again note that here* $\left\|\overrightarrow{GH}\right\| + \left\|\overrightarrow{HI}\right\| + \left\|\overrightarrow{IJ}\right\| \neq \left\|\overrightarrow{GJ}\right\|$.	

Magnitudes of non-parallel vectors cannot be added and subtracted by the usual rules of arithmetic.

Example 4

David Beckham kicked the ball from A to B, B to C and then from C to D.



(a) Express each of the following as a column vector.

$(i)\ \overrightarrow{AB},\quad (ii)\ \overrightarrow{BC},\quad (iii)\ \overrightarrow{CD},\quad (iv)\ \overrightarrow{AD}.$

(b) State whether the following statement is true or false.

(i) The sum of the magnitude $\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|+\left|\overrightarrow{CD}\right|$ represents the ***total distance*** traveled by the ball.

(ii) The vector $\overrightarrow{AD}$ represents the ***displacement*** of the ball.

(iii) $\left|\overrightarrow{AD}\right|=\left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|+\left|\overrightarrow{CD}\right|$?

(iv) $\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$?

Answers:

(a) $(i)\ \overrightarrow{AB}=\begin{pmatrix}-4\\3\end{pmatrix},(ii)\ \overrightarrow{BC}=\begin{pmatrix}-2\\-3\end{pmatrix},(iii)\ \overrightarrow{CD}=\begin{pmatrix}1\\-4\end{pmatrix},(iv)\ \overrightarrow{AD}=\begin{pmatrix}-5\\-4\end{pmatrix}$

(b) *(i) True*

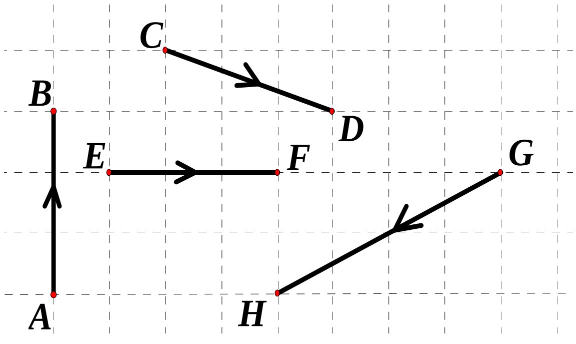
(ii) True

(iii) False because $\left|\overrightarrow{AD}\right|=\sqrt{5^2+4^2}=\sqrt{41},\ \left|\overrightarrow{AB}\right|+\left|\overrightarrow{BC}\right|+\left|\overrightarrow{CD}\right|=5+\sqrt{13}+\sqrt{17}$

(iv) True $\overrightarrow{AD}=\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}=\begin{pmatrix}-5\\-4\end{pmatrix}$

Example 5

If vectors $\overrightarrow{AB},\ \overrightarrow{CD},\ \overrightarrow{EF}$ *and* $\overrightarrow{GH}$ are added together, find the magnitude of the **resultant vector**.



Solution :

$\overrightarrow{AB}+\overrightarrow{CD}+\overrightarrow{EF}+\overrightarrow{GH}$

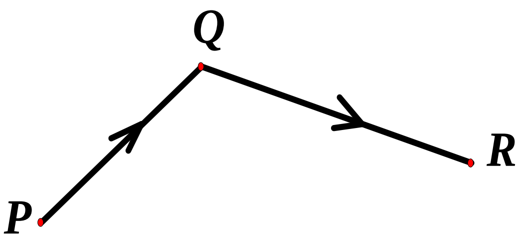
$=\begin{pmatrix}0\\3\end{pmatrix}+\begin{pmatrix}3\\-1\end{pmatrix}+\begin{pmatrix}3\\0\end{pmatrix}+\begin{pmatrix}-4\\-2\end{pmatrix}=\begin{pmatrix}2\\0\end{pmatrix}$

Magnitude of the resultant vector $=\sqrt{2^2+0^2}=2$.

Example 6

The diagram shows two vectors $\overrightarrow{PQ}$ and $\overrightarrow{QR}$.
Given that $\left|\overrightarrow{PQ}\right| = 3$, $\left|\overrightarrow{QR}\right| = 4$ and angle PQR is 120º,

find the magnitude of the resultant vector $\overrightarrow{PR}$.



Solution :

The vectors are not expressed in column vectors. To solve the problem, we need to use scale drawing or cosine rule.

$$\left|\overrightarrow{PR}\right|^2 = \left|\overrightarrow{PQ}\right|^2 + \left|\overrightarrow{QR}\right|^2 - 2\left|\overrightarrow{PQ}\right|\left|\overrightarrow{QR}\right|\cos 120^\circ$$

$$\left|\overrightarrow{PR}\right| = \sqrt{3^2 + 4^2 - 2(3)(4)\cos 120^\circ} = \sqrt{37} \approx 6.08$$

Example 7

Two forces each of 3 **N** and 4 **N** respectively are exerted on an object. Jane claims that the resultant force on the object must be 7 **N**. Comment.

Solution :

We need to consider the directions in which the forces are applied.

The magnitude of the resultant force can be computed using cosine rule (or scale drawing)
If the forces are acting in the same direction, the angle between the two forces is 180°,

The magnitude of the resultant force is $= \sqrt{3^2 + 4^2 - 2(3)(4)\cos 180^\circ} = \sqrt{25 + 24} = 7$.
If they are acting in the opposite directions, the angle between the two forces is 0°,

The magnitude of the resultant force is $= \sqrt{3^2 + 4^2 - 2(3)(4)\cos 0^\circ} = \sqrt{25 - 24} = 1$

If they are acting in any other directions such that the angle θ *between the two forces varies between 0° and 180° , the resultant force will vary between 1 N to 7 N.*

(E) Product of a Scalar and a Vector

For scalar $k > 0$, the scalar multiple $k\underset{\sim}{a}$ is a vector k times bigger than $\underset{\sim}{a}$ and in the ***same*** direction.
For scalar $k < 0$, the scalar multiple $k\underset{\sim}{a}$ is a vector k times bigger than $\underset{\sim}{a}$ and is in ***opposite*** direction.

eg. If $\overrightarrow{AB} = -8\mathbf{p} + 6\mathbf{q}$ and CD is parallel to AB and is half the size of AB, we can express vector $\overrightarrow{CD}$ as a ***scalar multiple*** of vector $\overrightarrow{AB}$, $\overrightarrow{CD} = \frac{1}{2}\overrightarrow{AB} = -4\mathbf{p} + 3\mathbf{q}$.

Properties of Scalar Multiples:

$(i)\ k\left(h\underset{\sim}{\mathbf{a}}\right) = \left(kh\right)\underset{\sim}{\mathbf{a}}$ $\quad (ii)\ k\left(\underset{\sim}{\mathbf{a}} + \underset{\sim}{\mathbf{b}}\right) = k\underset{\sim}{\mathbf{a}} + k\underset{\sim}{\mathbf{b}}$ $\quad (iii)\left(k + h\right)\underset{\sim}{\mathbf{a}} = k\underset{\sim}{\mathbf{a}} + h\underset{\sim}{\mathbf{a}}$

Physical examples of scalar multiples of vectors are
displacement = time × velocity where displacement and velocity are vectors, time is a scalar.
force = mass × acceleration where force and acceleration are vectors, mass is a scalar.

Example 8

Given that $\overrightarrow{AB} = 9\overrightarrow{CD}$, write down two facts about vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Solution :

Fact 1 : The two vectors have the same direction. Fact 2 : The magnitude of $\overrightarrow{AB}$ is 9 times that of $\overrightarrow{CD}$.

Example 9

The quadrilateral $ABCD$ is such that $\overrightarrow{AB} = 2\mathbf{p}$, $\overrightarrow{BC} = \mathbf{q}$ and $\overrightarrow{CD} = -3\mathbf{p}$. What is the special name given to the quadrilateral $ABCD$? *Answer : Trapezium*

(F) Position Vector

The position vector of a point P relative to the origin O is denoted by $\overrightarrow{OP}$.

If a point P has coordinates $(-2, 9)$ then $\overrightarrow{OP} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$.

All vectors can be expressed in terms of position vectors eg : $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$, $\quad \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

Example 10

(a) Express $\overrightarrow{RP} - \overrightarrow{TS} + \overrightarrow{SR} - \overrightarrow{SP} + \overrightarrow{TQ}$ as a single vector.

Solution :
One way to tackle the above problem is to "decompose" the vectors into their position vectors.

$$\overrightarrow{RP} - \overrightarrow{TS} + \overrightarrow{SR} - \overrightarrow{SP} + \overrightarrow{TQ} = \left(\overrightarrow{OP} - \overrightarrow{OR}\right) - \left(\overrightarrow{OS} - \overrightarrow{OT}\right) + \left(\overrightarrow{OR} - \overrightarrow{OS}\right) - \left(\overrightarrow{OP} - \overrightarrow{OS}\right) + \left(\overrightarrow{OQ} - \overrightarrow{OT}\right) = \overrightarrow{OQ} - \overrightarrow{OS} = \overrightarrow{SQ}$$

(b) Three points A, B and C have position vectors given by $\mathbf{p} - 4\mathbf{q}$, $2\mathbf{p} - \mathbf{q}$ and $4\mathbf{p} + 5\mathbf{q}$ respectively. Using vector method, show that the three points are collinear.

***Solution* :**

If three points lie on a straight line, they are said to be **collinear**. To prove it, we need to express say vector AB as a ***scalar multiple*** of say BC.

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (2\mathbf{p} - \mathbf{q}) - (\mathbf{p} - 4\mathbf{q}) = \mathbf{p} + 3\mathbf{q}$$

$$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = (4\mathbf{p} + 5\mathbf{q}) - (2\mathbf{p} - \mathbf{q}) = 2\mathbf{p} + 6\mathbf{q} = 2(\mathbf{p} + 3\mathbf{q}) = 2\overrightarrow{AB}$$

Since $\overrightarrow{BC} = 2\overrightarrow{AB}$, *the three points A, B and C are collinear.*

Example 11

Given that $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 9 \end{pmatrix}$, $\overrightarrow{BC} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$, $\overrightarrow{CD} = \begin{pmatrix} n \\ 13 \end{pmatrix}$.

(a) Express $2\overrightarrow{AB} + 5\overrightarrow{BC}$ as a column vector and hence find its magnitude.

(b) Given that $\overrightarrow{CD}$ is parallel to $\overrightarrow{AB}$, find the value of n**.**

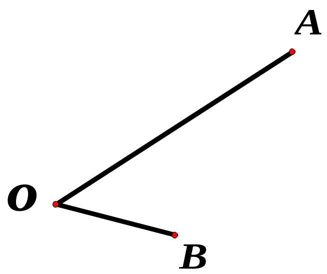
Solution:

(a) $2\overrightarrow{AB} + 5\overrightarrow{BC} = 2\begin{pmatrix} -2 \\ 9 \end{pmatrix} + 5\begin{pmatrix} 3 \\ 5 \end{pmatrix} = \begin{pmatrix} -4 \\ 18 \end{pmatrix} + \begin{pmatrix} 15 \\ 25 \end{pmatrix} = \begin{pmatrix} 11 \\ 43 \end{pmatrix}$, $\left|2\overrightarrow{AB} + 5\overrightarrow{BC}\right| = \sqrt{11^2 + 43^2} = \sqrt{1970} \approx 44.4$

(b) $\overrightarrow{CD} = m\overrightarrow{AB} \Rightarrow \begin{pmatrix} n \\ 13 \end{pmatrix} = m\begin{pmatrix} -2 \\ 9 \end{pmatrix} = \begin{pmatrix} -2m \\ 9m \end{pmatrix} \Rightarrow n = -2m \text{ and } 13 = 9m \Rightarrow n = -\frac{26}{9}$

Example 12

The diagram shows three points O, A and B where $A\ (8, 6)$ and $\overrightarrow{BC} = \begin{pmatrix} 12 \\ k \end{pmatrix}$. Given that BC is **parallel** to OA, find



(i) the value of k,

(ii) the ratio $\dfrac{\text{area of } \Delta OAB}{\text{area of } \Delta ACB}$.

GCE "O" N99/1/24

Solutions :

(i) BC parallel to OA implies $\overrightarrow{BC} = m\overrightarrow{OA}$,

$$\begin{pmatrix} 12 \\ k \end{pmatrix} = m\begin{pmatrix} 8 \\ 6 \end{pmatrix} = \begin{pmatrix} 8m \\ 6m \end{pmatrix} \Rightarrow m = \frac{3}{2} \Rightarrow k = 9$$

(ii)From (i) we get $\overrightarrow{BC} = \frac{3}{2}\overrightarrow{OA}$, *this implies* $\dfrac{|\overrightarrow{BC}|}{|\overrightarrow{OA}|} = \dfrac{3}{2}$

hence $\dfrac{\text{area of } \Delta OAB}{\text{area of } \Delta ACB} = \dfrac{|\overrightarrow{OA}|}{|\overrightarrow{BC}|} = \dfrac{2}{3}$, *triangles OAB and ACB are triangles with common height.*

Note:
*For triangles with common height, their **area ratio** is equal to their **base ratio**.*
*For similar triangles, their **area ratio** is equal to the **square** of their height ratio or base ratio or ratio of any corresponding sides.*

(G) The i, j notations

We introduce two **unit vectors**.

$\mathbf{i}$, $\mathbf{j}$ are unit vectors in the direction of the positive x – and y – axis respectively, i.e. $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$.

They are known as unit vector as their magnitude is 1, that is $|\ \mathbf{i}\ |=|\ \mathbf{j}\ |=1$.

All vectors can be expressed in terms of **i**, **j** ,

e.g, If $\overrightarrow{PQ}=\begin{pmatrix}2\\10\end{pmatrix}$, then $\overrightarrow{PQ}=\begin{pmatrix}2\\0\end{pmatrix}+\begin{pmatrix}0\\10\end{pmatrix}=2\begin{pmatrix}1\\0\end{pmatrix}+10\begin{pmatrix}0\\1\end{pmatrix}=2\mathbf{i}+10\mathbf{j}$. Generally $\overrightarrow{AB}=\begin{pmatrix}a\\b\end{pmatrix}=a\mathbf{i}+b\mathbf{j}$.

Example 13

Given $\overrightarrow{PQ}=-5\mathbf{i}+6\mathbf{j}$, $\overrightarrow{QR}=\mathbf{i}-3\mathbf{j}$ and $\overrightarrow{RS}=m\mathbf{i}+12\mathbf{j}$.

(i) Express $2\overrightarrow{PQ}-10\overrightarrow{QR}$ in terms of **i** and/or **j**.

(ii) Given that $\overrightarrow{RS}$ is **parallel** to $\overrightarrow{PQ}$, find the value of m.

(iii) Find $\left|\overrightarrow{PQ}\right|$. CFL

Solutions :

(i) $2\overrightarrow{PQ}-10\overrightarrow{QR}=2\begin{pmatrix}-5\\6\end{pmatrix}-10\begin{pmatrix}1\\-3\end{pmatrix}=\begin{pmatrix}-10\\12\end{pmatrix}-\begin{pmatrix}10\\-30\end{pmatrix}=\begin{pmatrix}-20\\42\end{pmatrix}=-20\mathbf{i}+42\mathbf{j}$

Note that it is easier to work with column format than i,j notation. However, you must remember to convert the column format back to i, j format.

(ii) $\overrightarrow{RS}=n\overrightarrow{PQ}\Rightarrow\begin{pmatrix}m\\12\end{pmatrix}=n\begin{pmatrix}-5\\6\end{pmatrix}\Rightarrow\begin{pmatrix}m\\12\end{pmatrix}=\begin{pmatrix}-5n\\6n\end{pmatrix}$

Solving $m=-5n$ and $12=6n$ simultaneously, we get $m=-10$.

(iii) $\left|\overrightarrow{PQ}\right|=\sqrt{(-5)^2+(6)^2}=\sqrt{61}$.

Example 14

Two points P and Q have position vectors given by $\mathbf{i}+8\mathbf{j}$ and $7\mathbf{i}-4\mathbf{j}$ respectively. The point R divides the line PQ in the ratio 1 : 5. Find the coordinates of R. CFL

Solution :

Let the coordinates of R be (a, b)

$$\overrightarrow{PQ}=6\overrightarrow{PR}\Rightarrow\overrightarrow{OQ}-\overrightarrow{OP}=6\left(\overrightarrow{OR}-\overrightarrow{OP}\right)$$

$$\begin{pmatrix}7\\-4\end{pmatrix}-\begin{pmatrix}1\\8\end{pmatrix}=6\begin{pmatrix}a\\b\end{pmatrix}-6\begin{pmatrix}1\\8\end{pmatrix}$$

$$\begin{pmatrix}6\\-12\end{pmatrix}=\begin{pmatrix}6a-6\\6b-48\end{pmatrix}\Rightarrow\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}2\\6\end{pmatrix}$$

Therefore the coordinates of R is (2, 6)